The completeness theorem:proofs and refutations

In propositional logic (use only **reversible** rules):

$$\dfrac{\dfrac{\dfrac{\dfrac{X \vdash X}{X \vdash X, Y}W \qquad \dfrac{\dfrac{Y \vdash Y}{Y, Z \vdash X, Y}W}{Y \wedge Z \vdash X, Y}\vee_m}{X \vee (Y \wedge Z) \vdash X, Y}\wedge_a}{X \vee (Y \wedge Z) \vdash X \vee Y}\vee_m \qquad \dfrac{\dfrac{\dfrac{X \vdash X}{X \vdash X, Z}W \qquad \dfrac{\dfrac{Z \vdash Z}{Y, Z \vdash X, Z}W}{Y \wedge Z \vdash X, Z}\vee_m}{X \vee (Y \wedge Z) \vdash X, Z}\wedge_a}{X \vee (Y \wedge Z) \vdash X \vee Z}\vee_m}{X \vee (Y \wedge Z) \vdash (X \vee Y) \wedge (X \vee Z)}\wedge_a$$

$$\dfrac{\dfrac{\dfrac{?}{Z \vdash X, Y}}{Z \vdash X \vee Y}\vee_m \qquad \dfrac{\dfrac{Z \vdash Z}{Z \vdash X, Z}W}{Z \vdash X \vee Z}\vee_m}{Z \vdash (X \vee Y) \wedge (X \vee Z)}\wedge_a$$

Take a distribution of truth values δ such that $\delta(Z) = 1$, $\delta(X) = \delta(Y) = 0$: one gets $\delta(Z) = 1$ and $\delta((X \vee Y) \wedge (X \vee Z)) = 0$.

In predicate logic: do the same **but** a refutation might be infinite (due to the irreversibility of $\exists$).

Semantics: denotational semantics and GOI

Basic idea: a proof of $A \vdash B$ is a function from $\mathcal{A}$ to $\mathcal{B}$.

Denotational semantics: associate with every formula A some kind of mathematical structure $\mathcal{A}$ and with every proof π of A an element $[\pi]$ of $\mathcal{A}$ in such a way that

- if π and π' are two proofs/terms such that $\pi \twoheadrightarrow \pi'$ then $[\pi] = [\pi']$ **(invariant)**
- if $[\pi] = [\pi']$ and λ (resp. λ') is obtained from π (resp. π') by applying the same rule, then $[\lambda] = [\lambda']$ **(congruence)**
- there exists a formula A and two proofs/terms π, π' of A s.t. $[\pi] \neq [\pi']$ **(non trivial)**

Geometry of interaction: represent the calculus itself within a mathematical structure: there exists a formula EX (the execution formula) such that if π' is the normal form of π, then $EX([\pi]) = [\pi']$.

"External" semantics

It is interesting for several reasons:

- it alows to *understand* internal structure: Cartesian Closed Categories, Scott D_∞,...

- it allows to reveal hidden structure and to produce new objects: Linear Logic, proof-nets.

- it allows to address precise technical questions, often difficult ones, and the efforts we do to answer them make us advance in the understanding of the objects (proofs/terms/computational processes) we deal with:
  - faithfullness/separation/injectivity
  - full completeness/surjectivity
  - full abstraction.

The main questions

Full completeness (surjectivity): A model is fully complete (surjective) w.r.t. a proof/term system when all the inhabitants of the interpretation of a formula are interpretations of a proof.

Faithfullness (separation, injectivity): A model is faithfull (injective) w.r.t. a proof/term system when it realizes the syntactical theory (two different normal proofs/terms are interpreted by two different objects in the model).

Full abstraction: A model is fully abstract w.r.t. an observational equivalence of a proof/term system when it realizes the observational equivalence.

“Internal” semantics

Basic idea: proofs and refutations are objects of the same nature. $\mathcal{A}=$ set of “proofs” or “refutations” of $A=$ set of “objects of type A satisfying some property” (every proof of A has to be an element of $\mathcal{A}$). A difficult point is to find the “abstract/mathematical” property characterizing these objects.

First (?) example of this kind of interpretation: the strong normalization proof of Gödel’s system T (Tait) and Girard’s system F. More recently, this approach led to Krivine’s realizability and Girard’s ludics.

Restating the main questions in terms of “internal semantics”, that is $\mathcal{A}=$ set of (cut-free and η-expanded) “para-proofs” of A, and $[\pi_A]=$ cut-free and η-expanded form of π_A:

- injectivity/faithfullness/separation: Böhm’s theorem: one can define a partial order on $\beta\eta$-equivalence classes of proofs (i.e. a T_0-topology)

- surjectivity/full completeness: “If $\pi_A \in \mathcal{A}$ (i.e. if π_A is a para-proof of A) and if π_A is not “para”, then π_A is a proof of A” (only example: Girard’s ludics)

- full abstraction: we need a notion of observational equivalence (values)

- what about “internal” GOI ?

External/internal separation: Injectivity/Böhm theorem

(M. Pagani and L. Tortora de Falco, Roma)

[Tortora de Falco03]: studies the question of injectivity for coherent and relational semantics

	set-based coherent semantics	multiset-based coherent semantics	relational semantics
$MELL$	NO	NO	? (yes)
$MELL \backslash \{?W\}$	? (yes)	? (yes)	? (yes)
LL_{pol}	? (yes)	? (yes)	? (yes)
$?\wp LL$	? (yes)	YES	YES

Open question: In [Laurent03], injectivity is proven w.r.t. "polarized games" for LL_{pol}. Is it also the case for coherent semantics (is there a way to deduce this from Laurent's result ?) This is related to the question of "uniformity" of coherent semantics (game semantics "is not uniform").

[Pagani03]: Towards Böhm theorem for the simply typed λ-calculus in a purely syntactical framework (no models).

Full abstraction and stability: coherent semantics for PCF

(L. Paolini, Torino)

Many people have studied how to extend a language in order to obtain the full-abstraction with respect to some models.

[Plotkin75]: the Scott domain model of PCF is fully abstract w.r.t. PCF extended with a parallel-if operator.

[Jim and Meyer76]: they consider the particular case of stable functions and show some negative results.

[Paolini03]: the coherent space semantics is fully abstract w.r.t. PCF extended with a Gustave-like operator and a new control operator test.

Open Questions:

1) Is it possible to refine the notion of stable function in order to rule out test-like functions?

2) What relations are there between the new language and strongly stable functions?

Full abstraction/full completeness and game models

(P. Di Gianantonio, Udine)

Game semantics has been used (introduced ?) to prove full abstraction and full completeness for various typed and untyped languages (most notably to answer the longstanding question of full abstraction for PCF [Abramsky-Jagadeesan-Malacaria94, Hyland-Ong94]).

[Di Gianantonio01]: introduces a new category of games, the monotonic games, and prove full abstraction for the pure lazy λ-calculus (i.e. the lazy λ-calculus without constants).

[Di Gianantonio-Ong03]: gives a characterization of Lévy-Longo trees in terms of "games with persistence"

Current lines of research: 1) use games with persistence to obtain a model of **sistem F** (in [Abramsky-Jagadeesan03] a game model of F has been given, but the games they used are rather different)

2) use games to define an "abstract semantics" (in the style of [Malacaria-Hankin98] and [Abramsky-Ghiga-Murausky-Ong03]), this should apply to verification.

Combinatory algebras, λ-models, Geometry of Interaction

(Honsell and Lenisa, Udine)

Particle vs Wave Geometry of Interaction

There are two variants of "GOI category" $\mathcal{C}$:

- **Particle-style/dynamic**: tensor on $\mathcal{C}$ is a **coproduct**, ! is a **countable copower**.
  Composition in the GoI category can be understood by simulating the flow of a particle around a network.
  Instances: Girard's GoI, games à la [AJM96], LCAs of [Abramsky-L.99,00,01]

- **Wave-style/static**: tensor on $\mathcal{C}$ is a **product**, ! is a **countable power**.
  Composition in the GoI category is defined statically and globally.
  Instances: Abramsky-Mellies99 concurrent games, timeless games of Baillot-Danos-Ehrard-Regnier97.

Axiomatization of GOI and λ-models

[Abramsky-Haghverdi-Scott02] answers two questions:

1. define a general axiomatization of GOI

2. Of what is a given GOI interpretation a model (in the traditional sense) ?

Remember [Barendregt 1984]:

- A λ-model is a combinatory algebra which is (also) a wealkly extensional λ-algebra

- $\lambda \vdash t = u \iff t = u$ holds in all λ-models.

[Abramsky-Haghverdi-Scott02]:

- define a general axiomatic setting for both the "wave-style" and the "particle-style" GOI called a **Geometry of Interaction situation**: a traced symmetric monoidal category + extra structure (essentially, a strong monoidal endofunctor on the category, which is given by the countable co-power (resp. power) in particle-style (resp. wave-style) models)

- exhibit a construction allowing to associate with every GOI situation, successively: a weak linear category, a linear combinatory algebra, a combinatory algebra.

Linear combinatory algebras stands in the same relation to the Hilbert-style axiomatization of the $(!, \multimap)$-fragment of linear logic as standard combinatory algebras do to minimal logic (typed λ-calculus).

We can now answer question 2: a GOI interpretation gives rise to a (linear) combinatory algebra, then to a λ-model (with its equational theory).

Basic Settings for particle/wave Semantics

Abramsky suggested

- $(Rel, +)$ together with a **trace operator** as **basic setting** for **particle-style/dynamic semantics**.

- $(Rel, \times)$ together with (another) **trace operator** as **basic setting** for **wave-style/static (traditional) semantics**.

But: while $(Rel, +)$ has been widely studied, $(Rel, \times)$ and the connections between wave GoI and traditional semantics have not. In particular, the thesis that $(Rel, \times)$ is the basic setting for static semantics is not immediate, because $(Rel, \times)$ **escapes** Abramsky's **GoI axiomatization**.

Problems

- How does $(Rel, \times)$ fit in the general axiomatic pattern?
- What kind of wave models arise from $(Rel, \times)$?
- Do wave models induce a **richer/different class** of **untyped λ-theories** than **game** (**particle**) models?

Some Answers [Honsell-L.03]

- We introduce a **weaker GoI axiomatization** (**weak GoI situation**). This allows to capture the category of **pointed relations** Rel^*.

- The GoI LCAs induced by Rel^* are **pointed linear graph models**.

- We introduce a notion of **strict (weak) GoI Situation**, in order to recover the case of $(Rel, \times)$.

- The GoI LCAs induced by Rel are **strict linear graph models**, i.e. models of **restricted** λ-calculi, such as the λI-calculus and the $\lambda\beta_{KN}$-calculus of [HL99].

- We show that, differently from game models, the class of λ-theories induced by wave models is quite rich, including also **non-sensible** (strict) λ-theories, i.e. not all unsolvable of order 0 are equated.

Others (F. Barbanera-U. de' Liguoro)

The theme is "semantics" of mobile calculi by means of logic.

[Barbanera-de' Liguoro03]: A semantics of mobile calculi modeling ambients carring code from object oriented languages is obtained by means of a type assignment system. This gives a logical interpretation of the calculus, which is adequate and complete.