Track
Structural Proof-Theory and Computational Complexity

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PROTOCOLLO – Final meeting
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Outline

Goal

To characterize complexity classes by structural proof-theory:

- **derivations-as-programs**
- **cut elimination-as-evaluation**
- **formulae-as-types**

Methodology

To limit the cut-elimination:

- Derivations structured into levels
- Duplication of specific derivations
- Duplication of derivations at level i implies:
  - New structure at levels $j > i$
  - Interaction only at levels $j > i$

Call it **Strong stratification**

The principles [Gir95]

- sub-recursive classes inside **LL**

ILAL	Elementary	Polynomial
$!(A_1 \otimes \cdots \otimes A_n) \multimap !A$	OK	$!A \multimap !B$
$!A \otimes !B \multimap !(A \otimes B)$	OK	Forbidden
$!A \multimap !A \otimes !A$	OK	OK
$!A \multimap !!A$	Forbidden	Forbidden
$!A \multimap A$	$!A \multimap \S A$	$!A \multimap \S A$
$!A \multimap \mathbb{I}$	OK	OK

- *soundness* and *completeness*: **ELL** w.r.t. elementary time; **LLL, LAL** w.r.t. polynomial time
- Let us call them *Light systems*

Outline

The meaning of characterizations

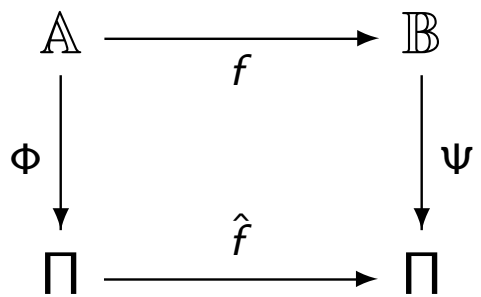
- A Light System **S** characterizes $\mathcal{C}$ if, **fixed** any $f \in \mathcal{C}$:
  - f can be represented by $\hat{f} \in \mathbf{S}$
  - normalizing $\hat{f}(\hat{n})$ costs $O(|\hat{f}(\hat{n})|^{e(d(\hat{f}(\hat{n})))})$:
    - $d(\hat{f}(\hat{n}))$ is the **depth** of $\hat{f}(\hat{n})$
    - e is some function;
  - $|\hat{n}|$ is linear in the value of n
  - $|\hat{f}|$ and $d(\hat{f}(\hat{n}))$ are **constants** w.r.t. $|\hat{n}|$
- **ELL**, **EAL**: e is a tower of exponentials
- **LLL**, **LAL**: e is a constant

Expressive power of fragments

- **ILAL** can also be:
  - **FDtime**$[2^{2^n}]$ sound and complete with **non constant depth** data-types as arguments [MaiMol02]
- **MLL**:
  - has a linear cut-elimination
  - can be **PDtime**$[n^k]$ sound and complete [Mair03,MairTer03]

Uniform representations [DL03]

- Separating, formally, the characterizations inside **ILAL**.
- $\mathbb{A} \xrightarrow{f} \mathbb{B} \in \mathcal{C}$ is **uniformly representable** if:



  - **commutes**
  - Π – set of arguments – has only cut-free derivations
  - $\Phi, \Phi^{-}, \Psi, \Psi^{-}$ are **Space**[ln] transducers

Canonical representations [DL03]

- $\mathbb{A} \xrightarrow{f} \mathbb{B} \in \mathcal{C}$ is **canonically representable** if
  - $\mathbb{A} \xrightarrow{f} \mathbb{B}$ is **uniformly representable**
  - Π – set of cut-free arguments – without $\forall$-left rules:
    - $\forall$-left rules may hide depth/dimension interplay
- **ILAL** is **FDtime**$[n^k]$ canonically sound and complete
- **ILAL** is **FDtime**$[2^{2^k}]$ uniformly, but not canonically, sound and complete.

Uniform representations [BDL04]

$\mathbf{FDtime}[n^k]$	$\supset$	$\mathbf{ILAL}_{\multimap\otimes}$	$\supset \{\text{polynomials on } \mathbb{N}\}$
$\mathbf{FDtime}[n^k]$	$\subset$	$\mathbf{ILAL}_{\multimap\otimes\forall}$	$\forall$ may link depth and dimension
$\mathbf{FDtime}[n^k]$	$=$	$\mathbf{ILAL}_{\multimap\otimes\overline{\mu}}$	linear recursive types don't ...
$\mathbf{FDtime}[n^k]$	$=$	$\mathbf{ILAL}_{\multimap\otimes\oplus\overline{\mu}}$	... as well as additives ...
$\mathbf{FDtime}[n^k]$	$=$	$\mathbf{ILAL}_{\multimap\otimes\overline{\forall}\overline{\mu}}$	... and $\forall$ on linear formulae (1)

- binary numerals in (1): $[01\epsilon] = \lambda 01\epsilon.0(\lambda 01\epsilon.1(\lambda 01\epsilon.\epsilon))$
- support conditional, but not iteration:

$$\begin{aligned} \mathtt{cond}\ \overline{2n}\ E\ O\ N &= E\ \overline{n} \\ \mathtt{cond}\ \overline{2n+1}\ E\ O\ N &= O\ \overline{n} \\ \mathtt{cond}\ \epsilon\ E\ O\ N &= N \end{aligned}$$

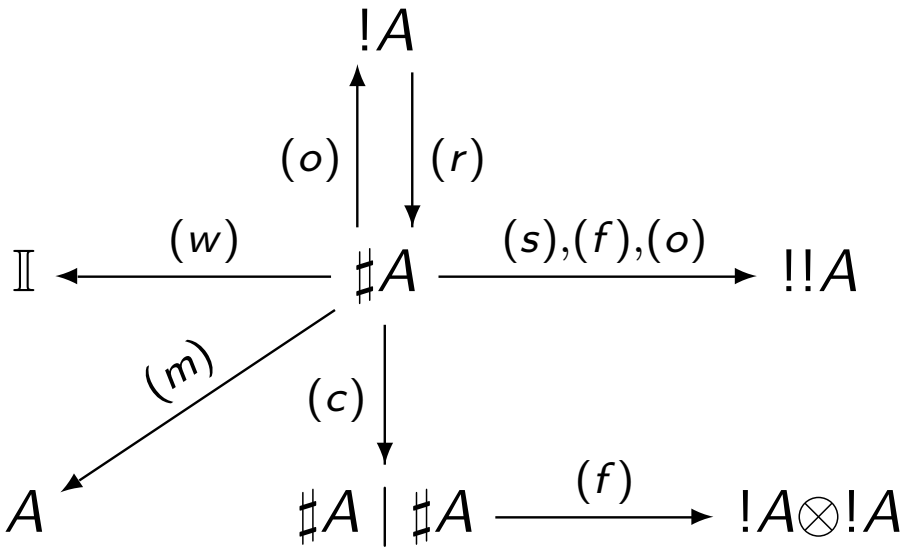
- $\mathtt{cond} = \lambda a.a(\lambda neo\epsilon.e\ n)(\lambda neo\epsilon.o\ n)(\lambda neo\epsilon.\epsilon)$

Outline

Initial table obsolete? [Maz04]

- $\vdash_{\mathbf{xUC}} \Sigma \parallel \Gamma$ with
  - $\Sigma = \alpha_1, \ldots, \alpha_n$
    - structural operations (contraction, ...)
    - α_i built with linear and structural symbols $\sharp, \flat$
  - $\Gamma = A_1, \ldots, A_n$
    - logical operations
    - A_i usual **LL** formula, possibly with §
- $! = \downarrow \sharp$, with $\downarrow$ positive shift
- $? = \uparrow \flat$, with $\uparrow$ negative shift
- $\uparrow$ / $\downarrow$ move formulae between Σ and Γ

Initial table obsolete? [Maz04]



Light system	Principles
ELL	forbid (o) and (r), *i.e.* $!A \neq \sharp A \ldots$
LLL	$\ldots +$ constraint on (f)
SLL	forbid (o) $+$ constraint on (f)

Initial table obsolete?

$$\begin{array}{rcl} \overline{\mathbf{LLL}} & = & \mathbf{ELL}\text{-DJ} \\ & + & \dfrac{\vdash ?\Gamma, \Delta}{\vdash ?\Gamma, \S\Delta}\ (\S) \\ & + & \text{affine !-boxes} \\ & + & \text{conditions on maximal exponential paths [Maz02]} \end{array}$$

$$\begin{array}{rcl} \mathbf{LLL}^* & = & \mathbf{LL} \\ & + & \dfrac{\vdash ?\Gamma, \Delta}{\vdash ?\Gamma, \S\Delta}\ (\S) \\ & + & \text{weights on formulae/rules recovering the meaning of} \\ & & \text{additive blocks in } \mathbf{LLL} \text{ [Maz03]} \end{array}$$

- both sound and complete w.r.t. $\mathbf{FDtime}[n^k]$
- $\mathbf{LLL}^*$ and $\mathbf{LLL}$ are equivalent w.r.t. provability

Outline

Light systems vs. recursive ones

$$\begin{array}{rcl} \textbf{HOLRR} & = & \text{linear affine } \lambda\text{-calculus} \\ & + & \text{higher-order ramified recursion [DLMR04]} \end{array}$$

- Sound and complete w.r.t. $\textbf{FDtime}[n^k]$
- $A ::= \textbf{Words} \mid A \otimes A \mid A \multimap A$
- *ramification*:
  - level on types that models *sorts*
  - $\neq$ stratification of *Light systems*
- *Recursion*: **binder of higher-order variables**
- *Polynomial normalization*:
  - the exponent of the polynomial depends on the nesting levels of the recursive operator
  - rewrites terms under a specific strategy
- *Completeness*: embedding **Ramified recurrence**

Light systems vs. recursive ones [Rov04]

- *Goal*: compositionally simulating recursive schemes
- The simulation can proceed bottom-up inside $\mathtt{T}^{\mathbf{w}}$:
  - $\mathtt{T}^{\mathbf{w}} \supset$ **ILAL**
  - !-boxes of $\mathtt{T}^{\mathbf{w}}$ may depend on arbitrary number of premises:
    - at most one can be !-discharged
    - the others must be §-discharged
- *Main question*: Is $\mathtt{T}^{\mathbf{w}}$ significantly more expressive than **ILAL**?

Outline

A hierarchy of complexity classes on rings [BlumShubSmale-end80]

- *Computational model*:
  - Cells of TM extended to contain "arbitrary" structure
  - Control operations on cells have unitary cost ...
    - multiplication on reals, included!
  - *Computation step*:
    - depend on the current state
    - *does not* depend on the cell being read

FDtime$[n^k]$ on $\mathbb{R}$ inside **ILAL** [BP04]

- *Goals*
  - To extend **ILAL** with δ-rules to simulate operations and relations on rings
  - To give alternative proof of polynomial time soundness and completeness of the model
- *Problem*
  - The effective expressivity of **ILAL** is to low
  - A "while" operator is required

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Lamping's algorithm

- It implements *optimal reductions à la Lévy* on λ-calculus
- It requires an oracle: maintain the consistence of graphs representing terms
- "No oracle" would mean *great speed-up*:

  most of the reductions may originate from the oracle
- **The λ-terms typeable in EAL do not require oracle!**

Automatic assessment of program complexity

- [CRDR03]
  - Let M be a pure λ-term
  - *Canonical abstract* **EAL**-*terms*: they highlight the structure that can duplicated
  - *Type inference*: it solves linear constraints to get **EAL** formulae as *principal* types
- [CM04]
  - Let M be typed Intuitionistic Logic (natural deduction)
  - *Maximal decoration of* M: symbolic distribution of as much boxes as possible
  - *Solving linear constraints*: they yield an **EAL** formula as type

Outline

- Informal workshop on 15 — 16, October 2003, Bertinoro.
  Participants: Coppola (Udine), Dal Lago (Bologna), Hoffman (Munich), Mairson (Boston - MA), Martini (Bologna), Mazza (Marseille), Ronchi Della Rocca (Torino), Roversi (Torino), Terui (Tokio), Tortora de Falco (Roma).
- Informal workshop on 28 — 29, June 2004, Torino.
  Participants: Coppo (Torino), Coppola (Udine), Dal Lago (Bologna), de'Liguoro (Torino), Dezani (Torino), Falzetta (Torino), Martini (Bologna), Masini (Verona), Mazza (Marseille), Paolini (Torino), Pedicini (Roma), Ronchi Della Rocca (Torino), Roversi (Torino).


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