INTERSECTION LOGIC

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joint work with Luca Roversi

the Curry-Howard isomorphism:
the logical foundation of typed languages

logical derivation | typed language

$\Pi : \sigma_1, \ldots, \sigma_n \vdash \sigma$ $\qquad$ $x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n \vdash \mathrm{term}(\Pi)^{\sigma}: \sigma$

- $\mathrm{term}(\Pi)^{\sigma}$ is the typed term codifying the proof Π
- the reduction rules of the language correspond to proof normalization steps

the Curry-Howard isomorphism: the type assignment case

logical derivations

typed language

$\Pi_1: \sigma_1, \ldots, \sigma_n \vdash \sigma$ $\quad$ $x_1:\sigma_1, \ldots, x_n:\sigma_n \vdash \text{term}(\Pi): \sigma$

$\Pi_2: \tau_1, \ldots, \tau_n \vdash \tau$ $\quad$ $x_1: \tau_1, \ldots, x_n: \tau_n \vdash \text{term}(\Pi): \tau$

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- Π_1, Π_2….. are proofs with the same schema
- Π is the schema of Π_1, Π_2…..
- term(Π) is an untyped term codifying the schema Π
- reduction rules correspond to proof schema normalization steps

the aim:

to find a logical foundation for IT
(Intersection Type Assignment System)

- IT has been designed in 1980 by Coppo-Dezani for increasing the typability power of Curry’s Type Assignment System

- IT characterizes strongly normalizing terms

- IT + ω (universal type) is a language for compact elements of domains

IT
(Intersection Type Assignment System)

$$(A)\ \frac{x{:}\sigma \in \Gamma}{\Gamma \ \vdash_{IT}\ x{:}\sigma}$$

$$(\wedge I)\ \frac{\Gamma \ \vdash_{IT} M{:}\sigma \quad \Gamma \ \vdash_{IT} M{:}\tau}{\Gamma \ \vdash_{IT} M{:}\sigma\wedge\tau}$$

$$(\wedge E^{l})\ \frac{\Gamma \ \vdash_{IT} M{:}\sigma\wedge\tau}{\Gamma \ \vdash_{IT} M{:}\sigma}$$

$$(\wedge E^{r})\ \frac{\Gamma \ \vdash_{IT} M{:}\sigma\wedge\tau}{\Gamma \ \vdash_{IT} M{:}\tau}$$

$$(\rightarrow I)\ \frac{\Gamma\cup\{x{:}\sigma\} \ \vdash_{IT} M{:}\tau}{\Gamma \ \vdash_{IT} \lambda x.M{:}\ \sigma \rightarrow \tau}$$

$$(\rightarrow E)\ \frac{\Gamma \ \vdash_{IT} M{:}\sigma \rightarrow \tau \quad \Gamma \ \vdash_{IT} N{:}\sigma}{\Gamma \ \vdash_{IT} MN{:}\tau}$$

the problem:

the logical interpretation of the rule:

$$(\wedge I) \quad \frac{\Gamma \mid\!-_{IT} M{:}\sigma \qquad \Gamma \mid\!-_{IT} M{:}\tau}{\Gamma \mid\!-_{IT} M{:}\ \sigma \wedge \tau}$$

where the term M plays the role of a modality

LJ

(implicative and conjuctive fragment of Intuitionistic Logic)

$$(A)\ \frac{\sigma \in \Gamma}{\Gamma \ |\!-\ \sigma}$$

$$(\wedge I)\ \frac{\Gamma \ |\!-\ \sigma \quad \Gamma \ |\!-\ \tau}{\Gamma \ |\!-\ \sigma \wedge \tau}$$

$$(\wedge E^{l})\ \frac{\Gamma \ |\!-\ \sigma \wedge \tau}{\Gamma \ |\!-\ \sigma}$$

$$(\wedge E^{r})\ \frac{\Gamma \ |\!-\ \sigma \wedge \tau}{\Gamma \ |\!-\ \tau}$$

$$(\rightarrow I)\ \frac{\Gamma \cup \{\sigma\} \ |\!-\ \tau}{\Gamma \ |\!-\ \sigma \rightarrow \tau}$$

$$(\rightarrow E)\ \frac{\Gamma \ |\!-\ \sigma \rightarrow \tau \qquad \Gamma \ |\!-\ \sigma}{\Gamma \ |\!-\ \tau}$$

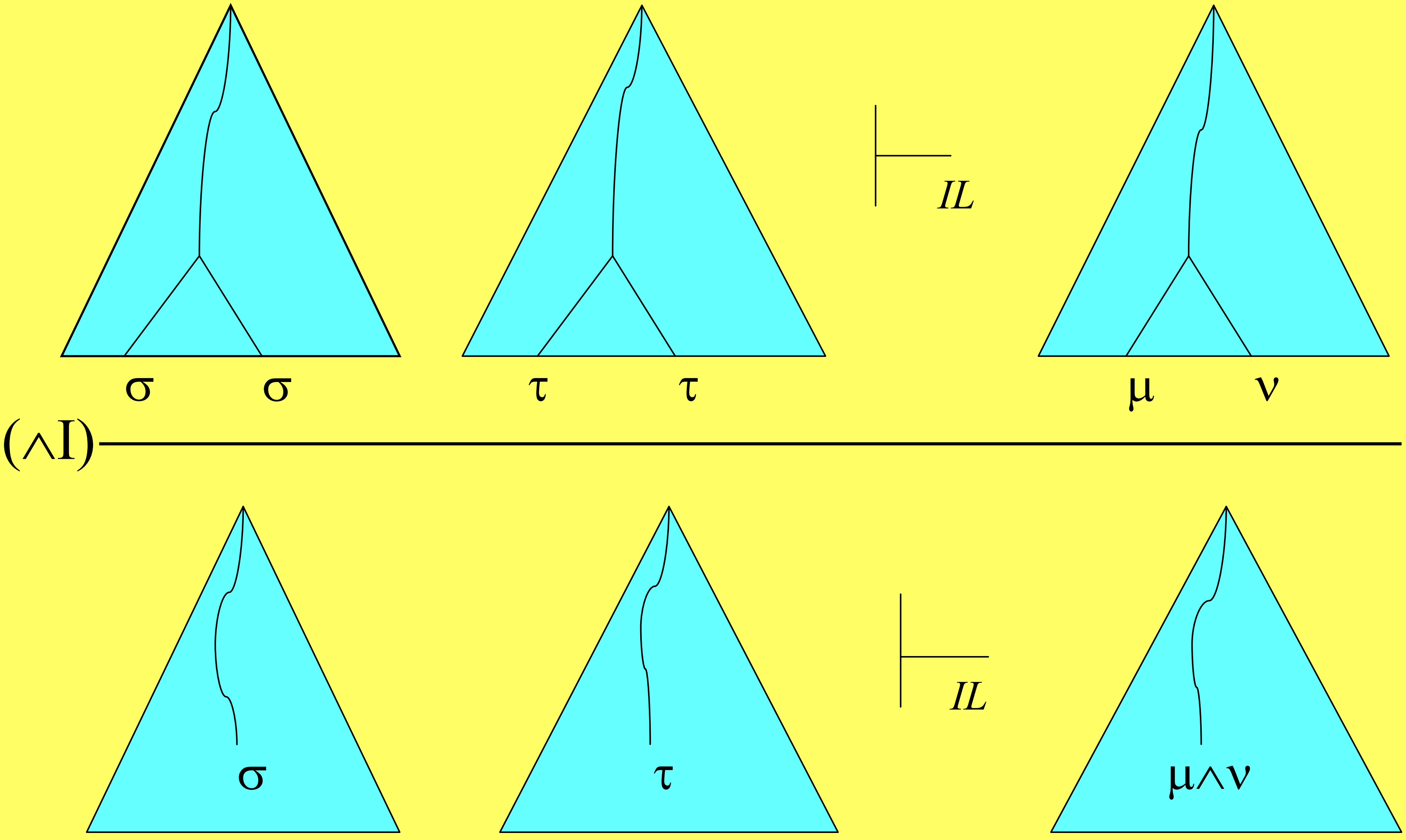
INTERSECTION LOGIC
IL

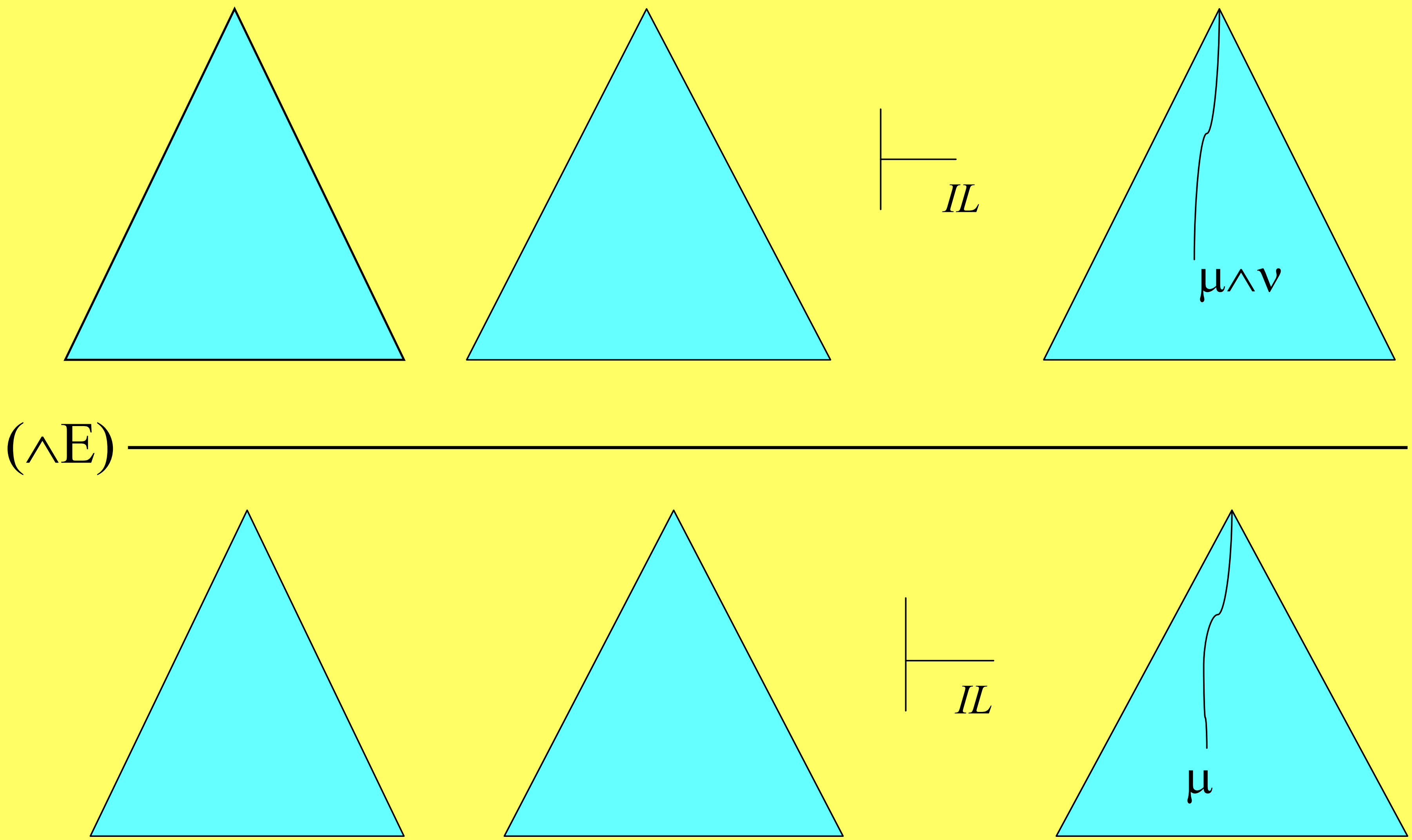
- formulae are trees of formulae of *LJ* (called kits)

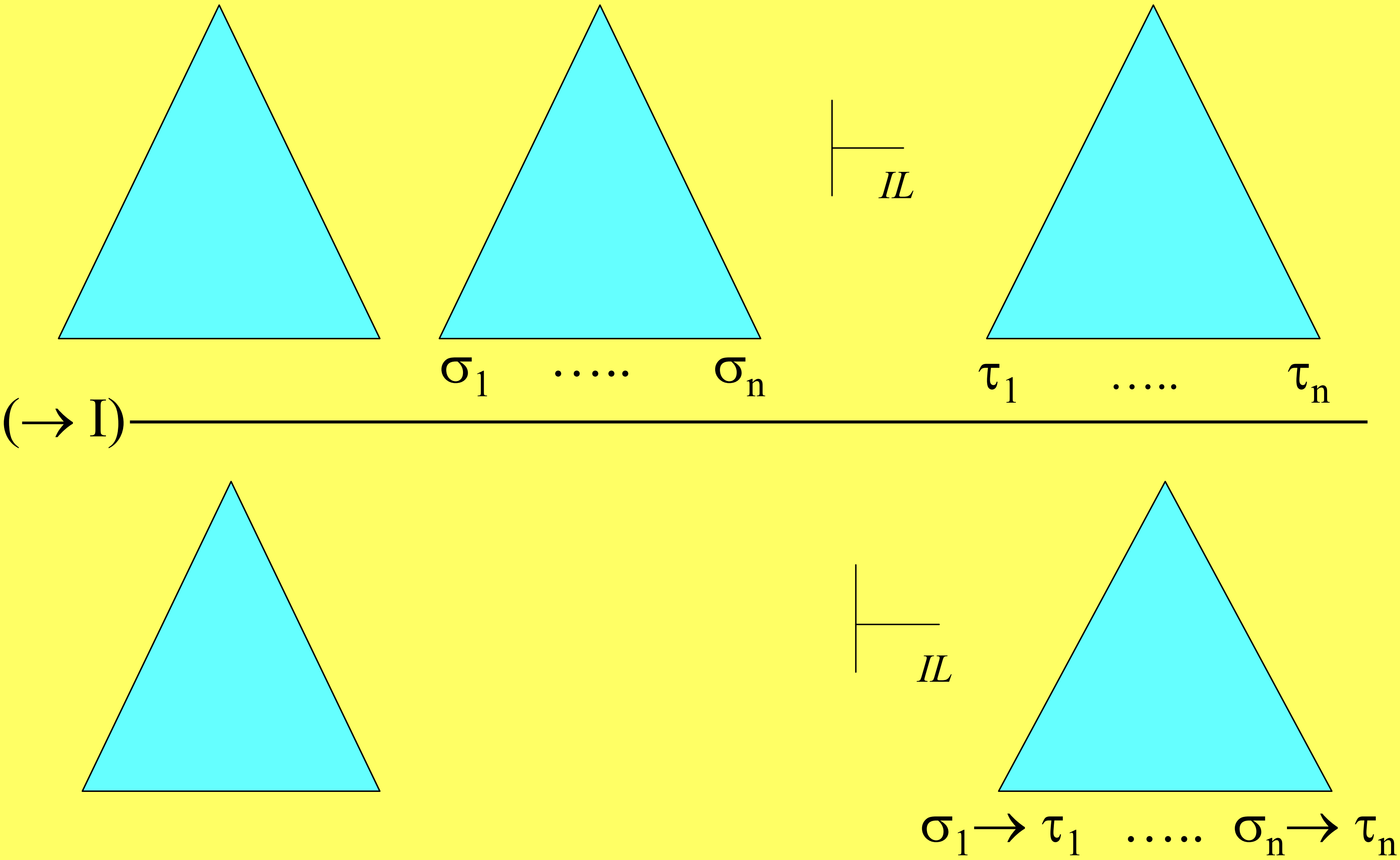
- the judgments are of the shape:

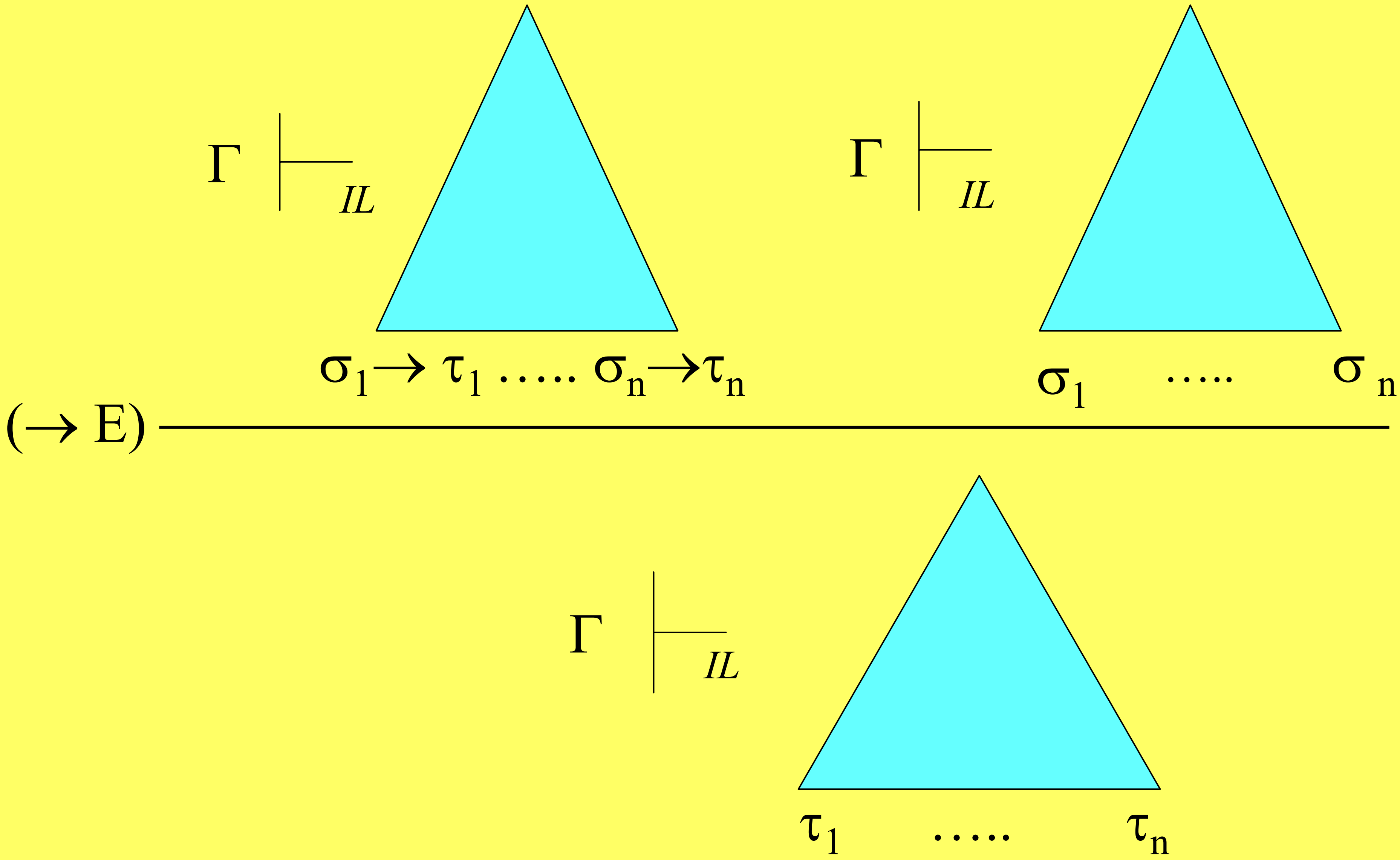
$$\{H_1, \ldots, H_n\} \vdash_{IL} H$$

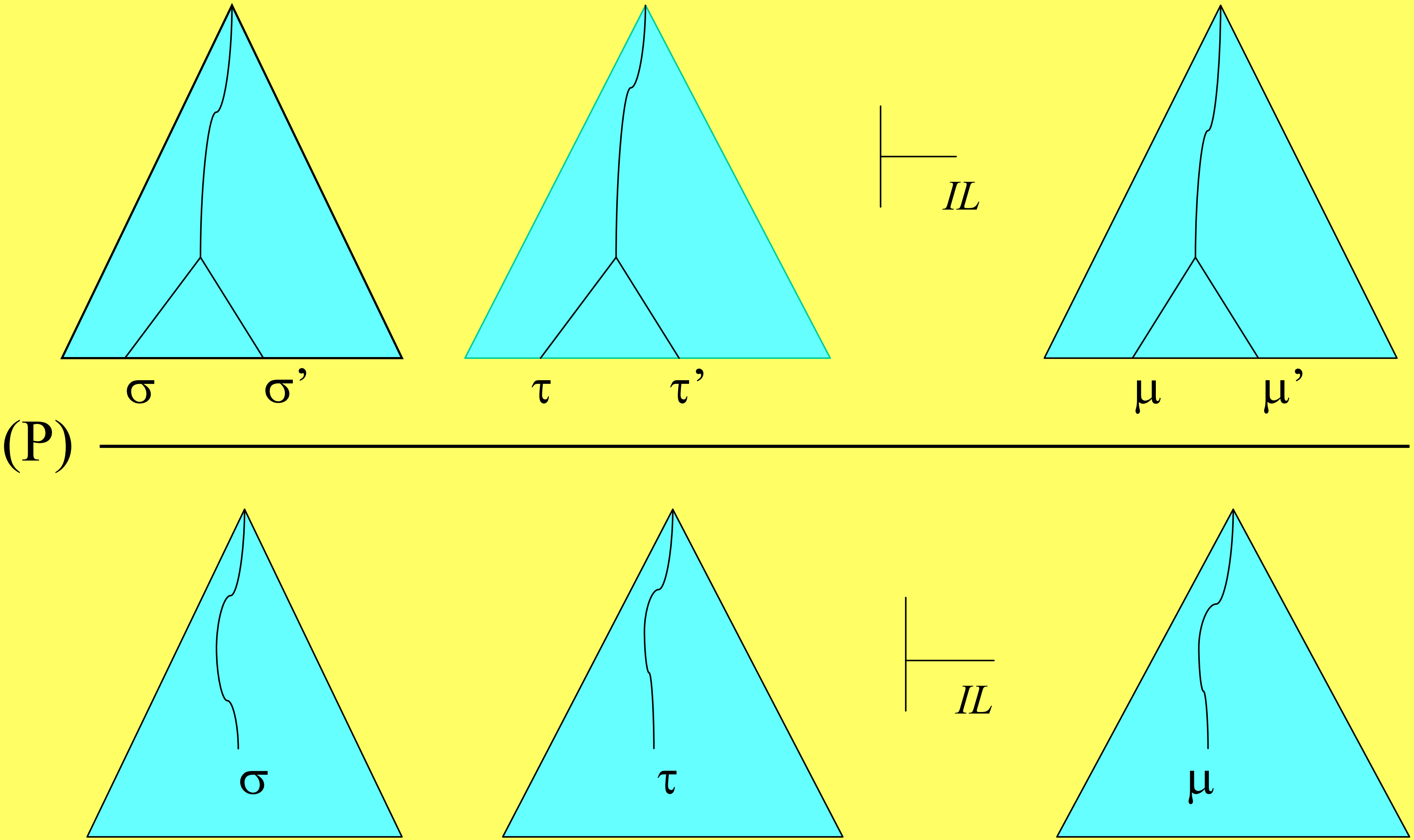
where H_i and H are kits of the same shape


σ σ
τ τ
IL
μ ν
(∧I)
σ
τ
IL
μ∧ν


IL
μ∧ν
(∧E)
IL
μ


⊢ IL
σ1 ….. σn
τ1 ….. τn
(→ I)
⊢ IL
σ1→ τ1 ….. σn→ τn

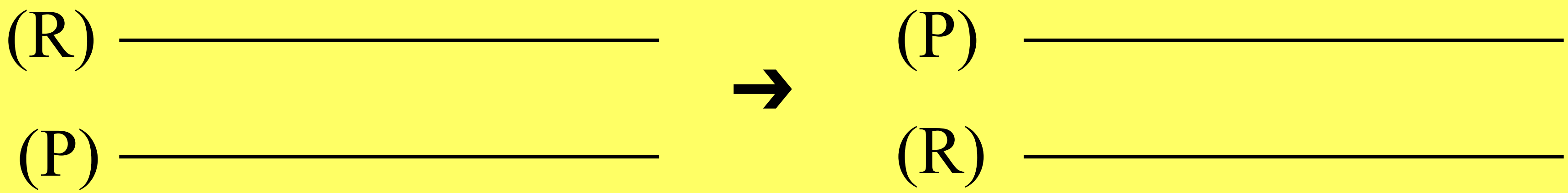

Γ ⊢ IL
σ1→ τ1 ….. σn→τn
Γ ⊢ IL
σ1 ….. σ n
(→ E)
Γ ⊢ IL
τ1 ….. τn

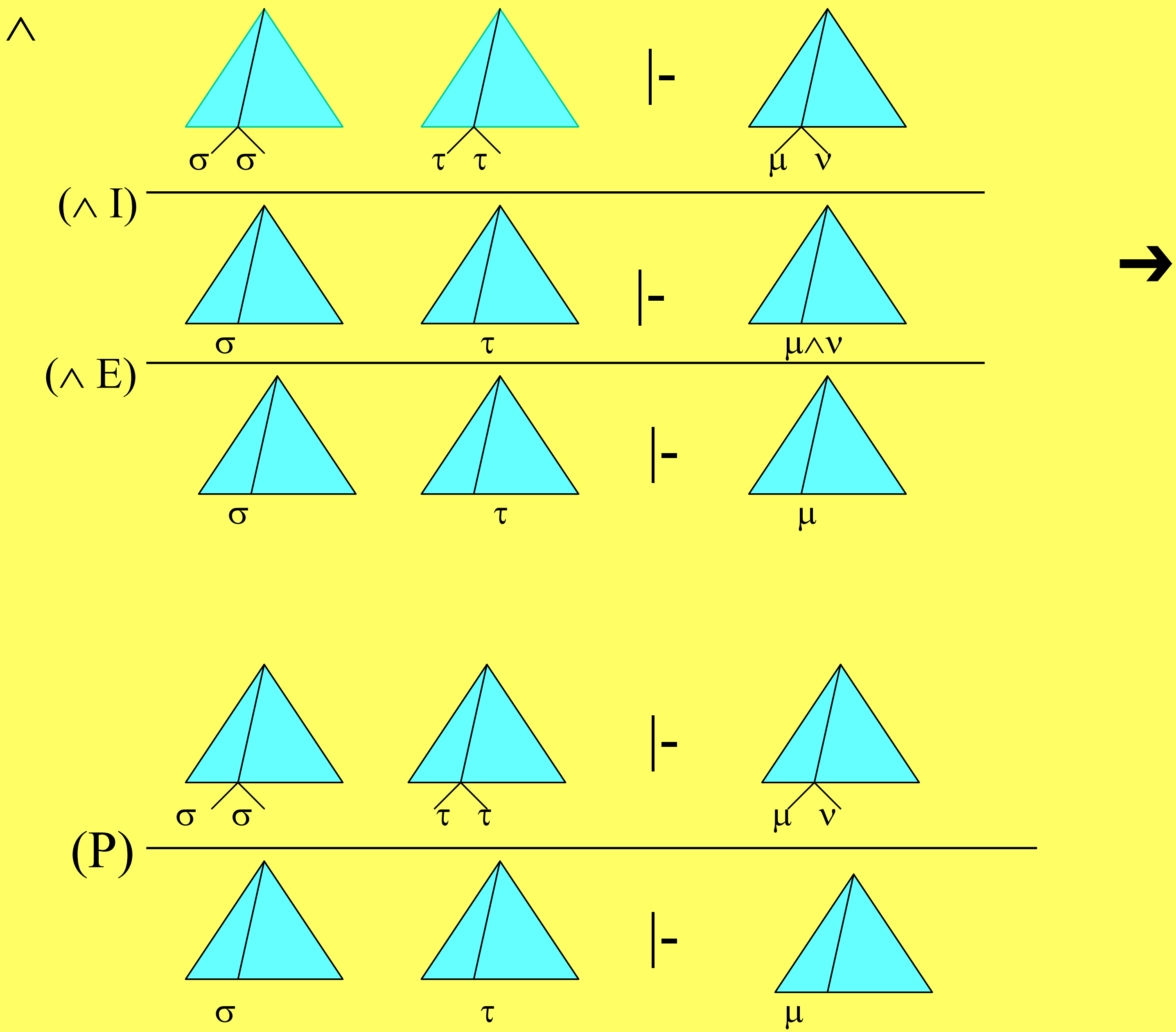

σ σ'
τ τ'
IL
μ μ'
(P)
σ
τ
IL
μ

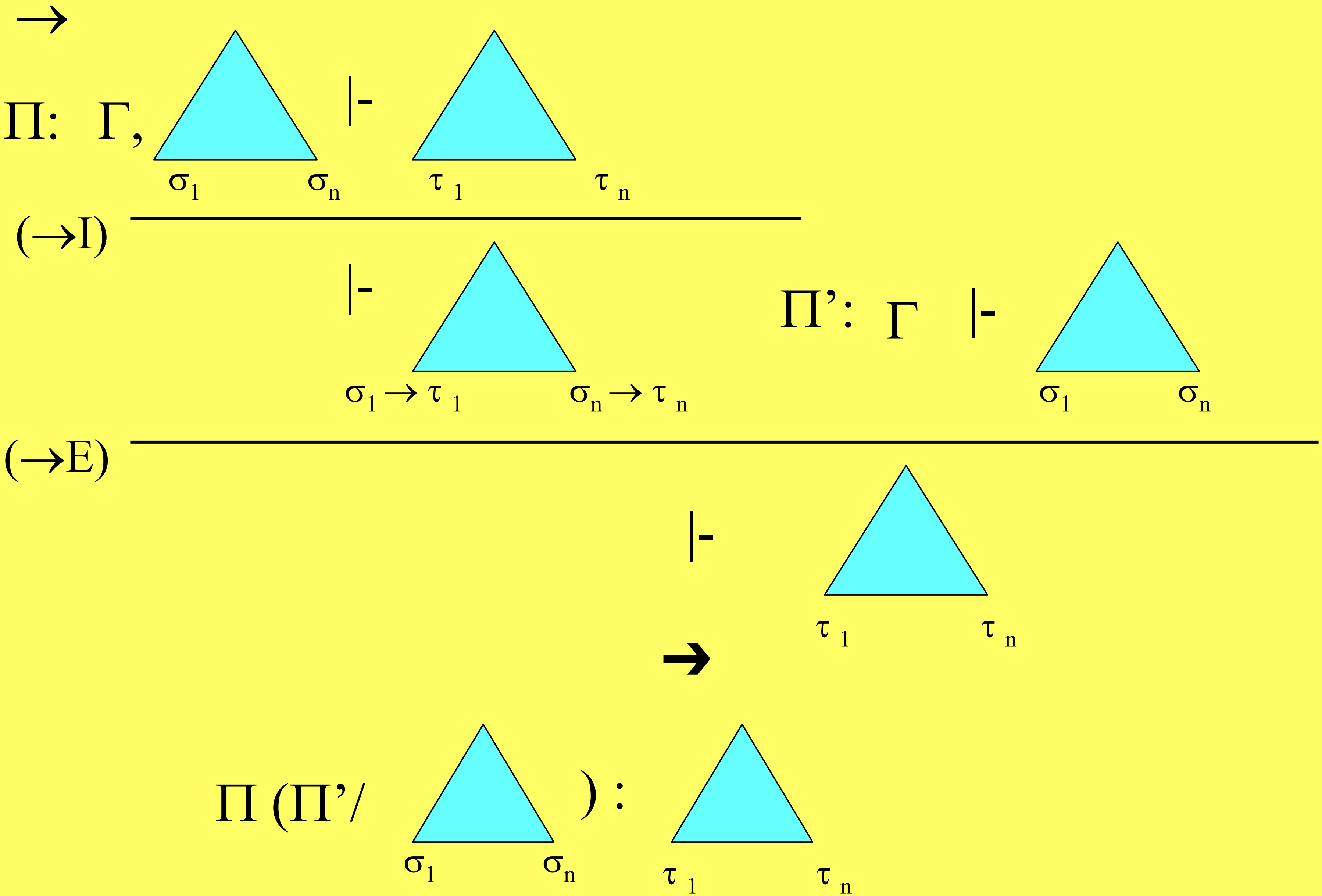
Derivations of *IL* (Intersection Logic) are the derivations built from the previous showed rules modulo some commutations of rules dealing with $\wedge$ on different paths

Theorem:
IL is strongly normalizing

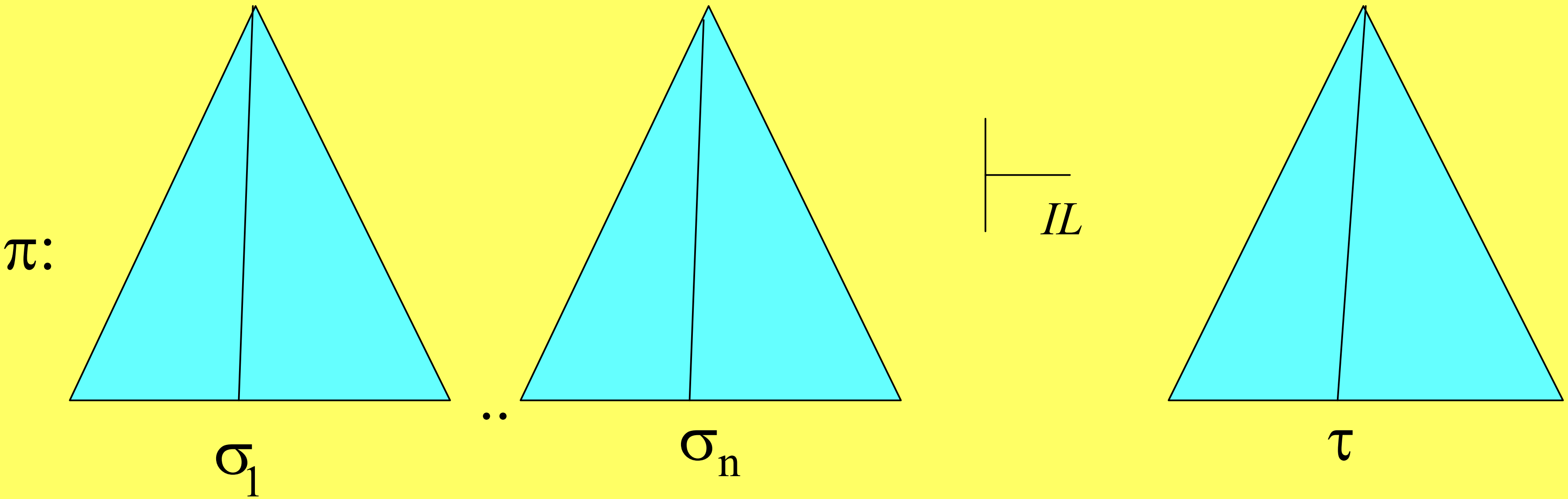
P-normalization step




∧
σ σ
τ τ
|-
μ ν
(∧ I)
σ
τ
|-
μ∧ν
(∧ E)
σ
τ
|-
μ
σ σ
τ τ
|-
μ ν
(P)
σ
τ
|-
μ


→
Π: Γ, |-
σ1 σn τ 1 τ n
(→I)
|-
σ1 → τ 1 σn → τ n
Π': Γ |-
σ1 σn
(→E)
|-
τ 1 τ n
➔
Π (Π'/) :
σ1 σn τ 1 τ n

Theorem:



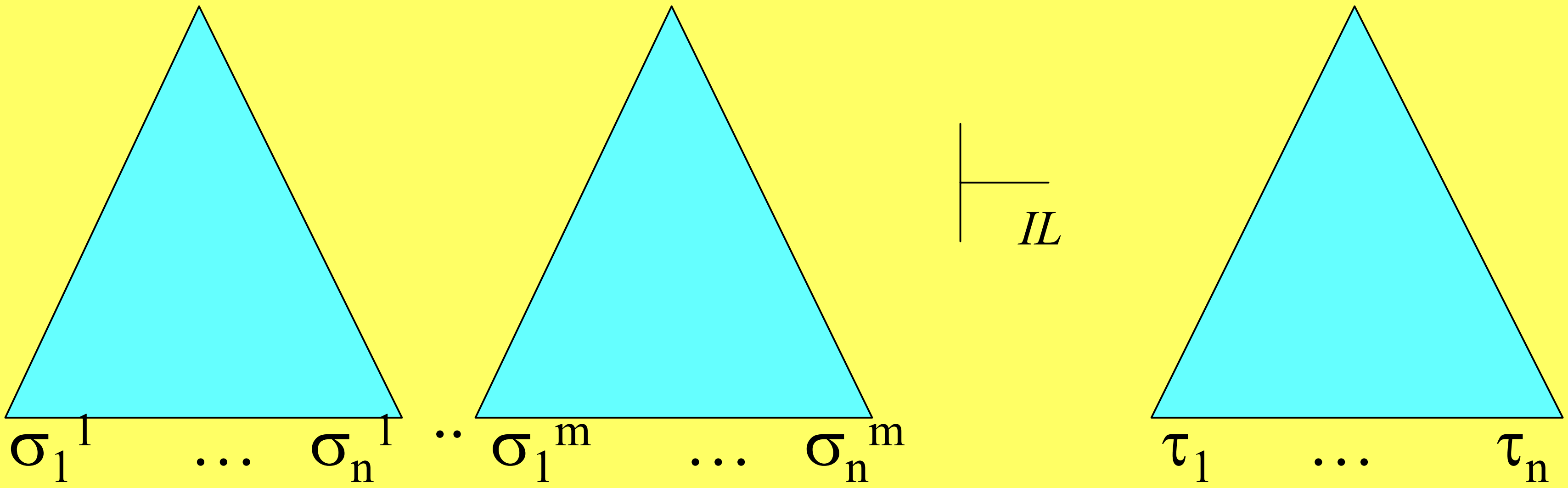
implies

$$\Pi: \{\sigma_1, \ldots, \sigma_n\} \vdash_{LJ} \tau$$

and π and Π have the same shape

Theorem:

π:



if and only if, for all i ($1 \leq i \leq n$)

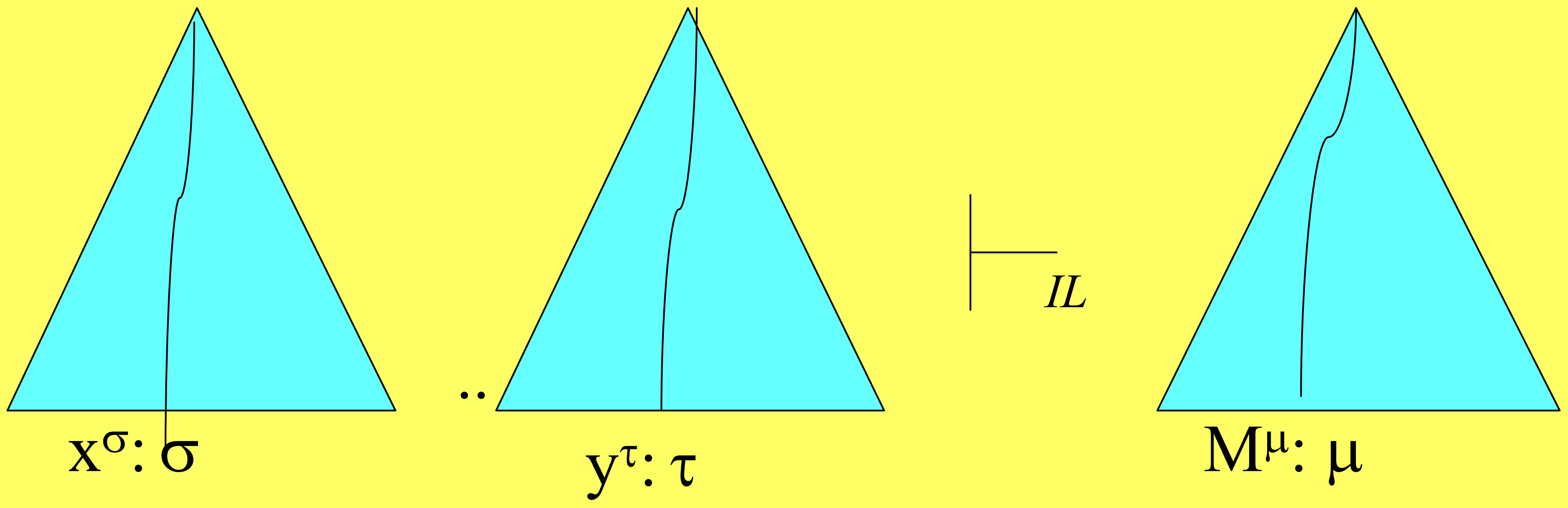
$$\Pi_i\text{: } \{x_1{:}\sigma_i^{\,1}, \ldots, x_n{:}\sigma_i^{\,m}\} \;\; \vdash_{IT} \; \text{term}(\pi){:}\tau_i$$

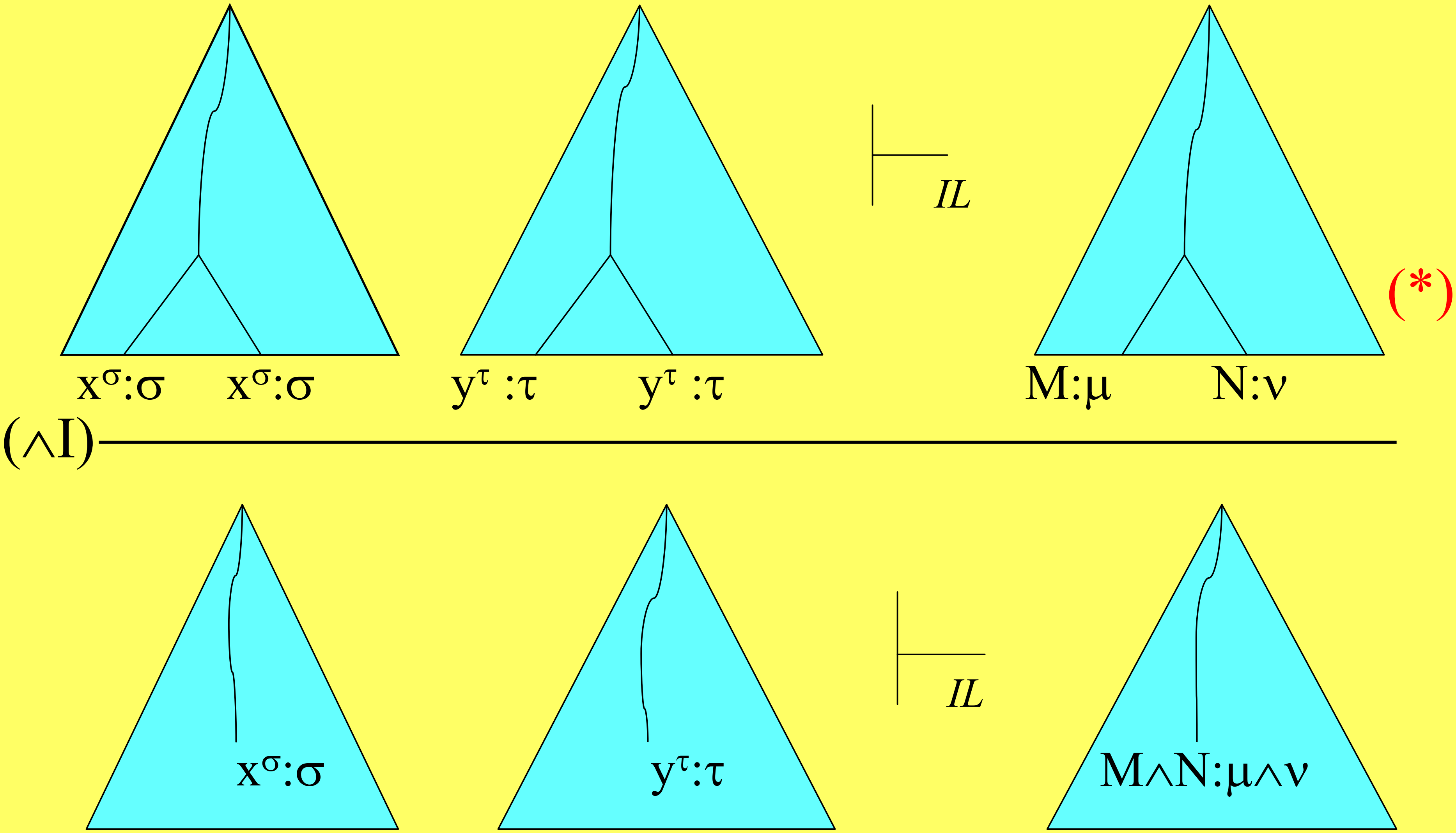
- $FV(M) \subseteq \{x_1, \ldots, x_n\}$
- π and Π_i have the same shape

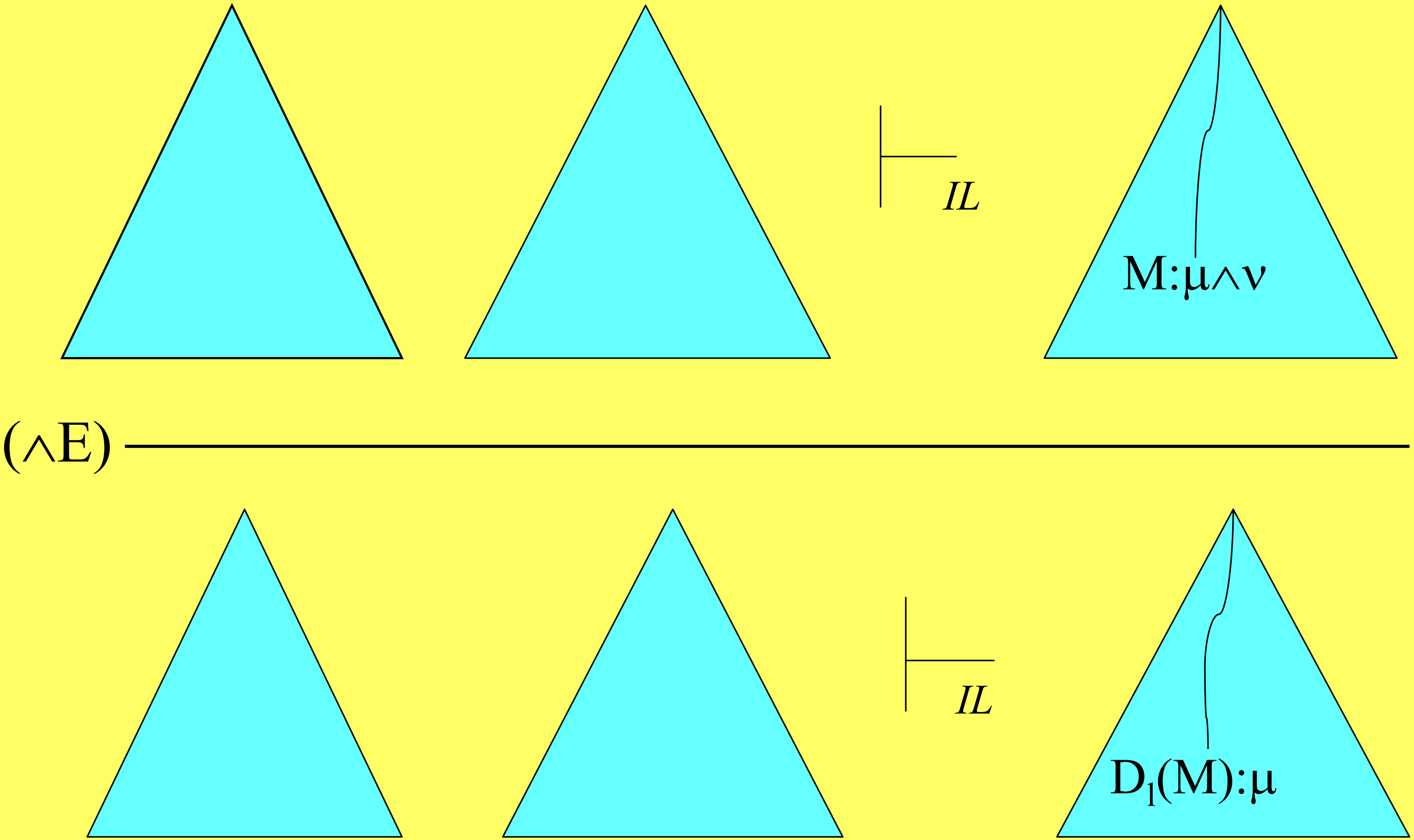
1 application of β-rule corresponds to:

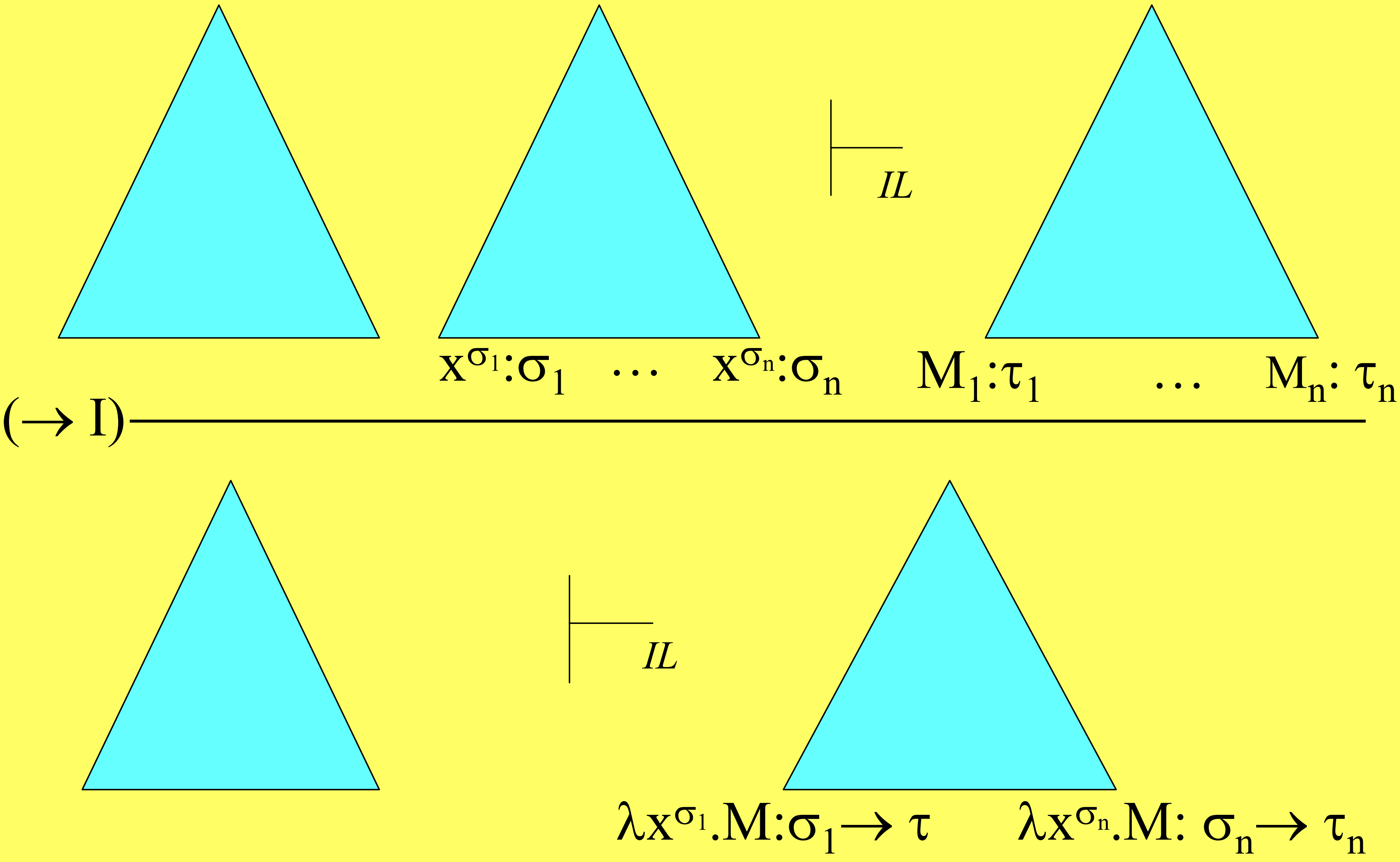
- 1 application of $(\rightarrow)$-normalization step
- $n \geq 0$ applications of $(\wedge)$-normalization steps
- $n \geq 0$ applications of (P)-normalization steps

Decorating IL




$x^\sigma:\sigma$ $x^\sigma:\sigma$
$y^\tau:\tau$ $y^\tau:\tau$
$\vdash_{IL}$
$M:\mu$ $N:\nu$
(*)
$(\wedge I)$
$x^\sigma:\sigma$
$y^\tau:\tau$
$\vdash_{IL}$
$M\wedge N:\mu\wedge\nu$


IL
M:μ∧ν
(∧E)
IL
D1(M):μ


$\vdash_{IL}$
$x^{\sigma_1}:\sigma_1 \quad \dots \quad x^{\sigma_n}:\sigma_n \qquad M_1:\tau_1 \quad \dots \quad M_n:\tau_n$
$(\rightarrow I)$
$\vdash_{IL}$
$\lambda x^{\sigma_1}.M:\sigma_1 \rightarrow \tau \qquad \lambda x^{\sigma_n}.M:\sigma_n \rightarrow \tau_n$

$\Gamma \vdash_{IL}$ $\qquad\qquad$ $\Gamma \vdash_{IL}$ $\qquad$ (**)

$M_1 : \sigma_1 \to \tau_1 \ldots M_n : \sigma_n \to \tau_n \qquad N_1 : \sigma_1 \quad \ldots \quad N_n : \sigma_n$

$(\to E)$ ——————————————————————————

$\Gamma \vdash_{IL}$

$M_1 N_1 : \tau_1 \quad \ldots \quad M_n N_n : \tau_n$

The typed language comes naturally with an erasure function E

definition of E:

………

$E(M \wedge N) = E(M)$ if $E(M) \equiv E(N)$, undefined otherwise

……….

(E is total, when applied to decoration terms)

formation rules of terms:

………

$M \wedge N$ is well formed if and only if $E(M) \equiv E(N)$

……….

Reduction rules (derived from the logic)

since the global behaviour of $\rightarrow$:

$$(\lambda x^{(\sigma\rightarrow\sigma)\rightarrow\sigma\rightarrow\sigma}.x)(\lambda y^{\sigma\rightarrow\sigma}.y) \wedge (\lambda x^{(\tau\rightarrow\tau)\rightarrow\tau\rightarrow\tau}.x)(\lambda y^{\tau\rightarrow\tau}.y)$$

must reduce in one step to

$$(\lambda y^{\sigma\rightarrow\sigma}.y) \wedge (\lambda y^{\tau\rightarrow\tau}.y)$$

Properties of typed terms

- free and bound variables have different names

- $(M^{\sigma} \wedge N^{\tau})^{\sigma \wedge \tau}$ implies $E(M) \equiv E(N)$ (erasure function)
(condition (*) assures that this is true also for bound variables)

- $(\lambda x^{\sigma}.M)$ and $(\lambda x^{\tau}.N)$ are subterms of R implies there are P and Q such that $P \wedge Q$ is a subterm of R and $(\lambda x^{\sigma}.M)$ and $(\lambda x^{\tau}.N)$ are corresponding subterms of P and Q respectively (condition **)

Reduction rules
(inherited from *IL*)

- $(\lambda x^{\sigma}.M[x_1^{\sigma},\ldots, x_n^{\sigma}])^{\sigma\to\tau}N^{\sigma} \to_{\beta} (M[N_1/x_1^{\sigma},\ldots, N_n/x_n^{\sigma}])^{\tau}$
  where $N_1,\ldots, N_n$ are copies of N such that $i \neq j$ implies $BV(E(N_i))$ and $BV(E(N_j))$ are disjoint and the renaming is made using fresh variables. (β-redex)

- $(D_l(M^{\sigma}\wedge N^{\tau})^{\sigma\wedge\tau})^{\sigma} \to_{\wedge} M^{\sigma}$ (∧-redex)

- $(D_r(M^{\sigma}\wedge N^{\tau})^{\sigma\wedge\tau})^{\tau} \to_{\wedge} N^{\tau}$ (∧-redex)

Extension to terms

$M \Rightarrow_\beta N$ if N is obtained from M by reducing in parallel all the β-redexes having the same erasure

$M \Rightarrow_\wedge N$ if N is obtained from M by reducing an ∧-redex

(there is not α-rule!)