On the number of provable formulas

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Cofin 2000 "Logica Lineare e oltre..."
Università Roma Tre, 11/02/2003

Overview

- Provable formulas of MLL
- Some combinatorical notions
- The negatively focalized calculus
- Splitting tensors
- A tree-algebra to compute the number of proof nets
- The number of derivations; an upperbound
- A heuristic lowerbound based on the number of derivations
- Conclusion and remarks

Formulas of MLL

The set of formulas of multiplicative linear logic (MLL) is as usual defined as the smallest set containing:

- a countable set of positive atoms $\alpha_1, \alpha_2, \ldots$
- respective negative atoms $\alpha_1^\perp, \alpha_2^\perp, \ldots$,
- and closed under par ($\parr$) and tensor ($\otimes$).

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Moreover negation $(-)^\perp$ is defined by

- $(\alpha)^\perp := \alpha^\perp$ and $(\alpha^\perp)^\perp := \alpha$;
- $(A \parr B)^\perp := A^\perp \otimes B^\perp$ and
- $(A \otimes B)^\perp := A^\perp \parr B^\perp$.

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Sequents are multisets of formulas

Rules of MLL

$$\frac{}{\vdash \alpha_i, \alpha_i^\perp}\,\mathsf{Ax} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \qquad \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes$$

In a proof we can assume the axioms to be different, implying that all conclusions are different. So the sequents are in fact sets of formulas rather than multisets.

Provable sequents of MLL

Lemma 1. *Suppose $\vdash \Gamma$ is derivable and contains $2n$ atoms, p pars, t tensors and consists of c conclusions. Then the following holds: $c+p = n+1$ and $t = n-1$. As a consequence $1 \leq c \leq n+1$. A provable formula ($c = 1$) has $p = n$ pars and $t = n-1$ tensors.*

Proof:

Let π be a proof of $\vdash \Gamma$ consisting of n atomic axioms, p pars, t tensors and c conclusions.

If π is an axiom the property holds.

If π ends by a $\parr$-rule two conclusions are replaced by only one, while p increases by one.

If π ends by a $\otimes$-rule, we know by induction hypothesis that $c_i+p_i = n_i+1$ and $t_i = n_i - 1$ while $n = n_1 + n_2$, $c = c_1 + c_2 - 1$, $p = p_1 + p_2$ and $t = t_1 + t_2 + 1$. So $c + p = c_1 + c_2 - 1 + p_1 + p_2 = n_1 + 1 + n_2 = n + 1$ and $t = t_1 + t_2 + 1 = n_1 - 1 + n_2 = n - 1$.

Provable formulas of MLL

By the lemma, we will find provable formulas only among the so-called balanced formulas: formulas with $2n$ atoms (pairwisely positive and negative), $p = n$ pars and $t = n - 1$ tensors.

Such a formula is given by a list of the $2n$ (positive and negative) atoms, moreover a binary tree with $2n - 1$ connectives, n of which are par while the remaining $n - 1$ are tensor.

Some combinatorical notions

- Choosing k out of n
- Catalan numbers
- The number of balanced formulas
- An equivalence relation on balanced formulas
- Expectation

Choosing k out of n

The number of ways to choose k objects out of a set of n distinct objects is given by the binomial coefficient

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

Catalan numbers

The number of distinct binary trees with $k+1$ leaves is given by the the k-th Catalan number

$$C_k := \frac{1}{k+1}\binom{2k}{k}$$

E.g. there are $C_3 = 5$ binary trees with 4 leaves:

$$((x \cdot x) \cdot x) \cdot x$$

$$(x \cdot (x \cdot x)) \cdot x$$

$$(x \cdot x) \cdot (x \cdot x)$$

$$x \cdot ((x \cdot x) \cdot x)$$

$$x \cdot (x \cdot (x \cdot x))$$

The number of balanced formulas

A balanced formula is a formula with the $2n$ atomic subformulas

$$\alpha_1, \alpha_1^\perp, \ldots, \alpha_n, \alpha_n^\perp$$

and with $p = n$ pars and $t = n - 1$ tensors.

Such a formula is given by a list of the $2n$ (positive and negative) atoms, moreover a binary tree with $2n - 1$ connectives ($2n$ leaves), n of which are pars while the remaining $n - 1$ are tensors. Hence there are

$$\binom{2n-1}{n} C_{2n-1} (2n)!$$

of them.

An equivalence relation on balanced formulas

Let us call two balanced formulas equivalent iff they differ only by a name or sign of an atom. To be precise,

$$\phi[a_i, a_i^\perp] \sim \phi[a_i^\perp, a_i]$$
$$\phi[a_i, a_i^\perp, a_j, a_j^\perp] \sim \phi[a_j, a_j^\perp, a_i, a_i^\perp]$$

Then each equivalence class consists of $2^n n!$ formulas. In fact, in the sequel we will quotient by this equivalence for most of the results.

The negatively focalized calculus

We will now consider the negatively focalized sequent calculus of MLL, notation $\mathsf{MLL}_{\mathsf{nf}}$.

There are two types of sequents:

- A positive sequent of length c is a set of c final-$\parr$-free formulas, notation $\vdash A_1, \ldots A_c$.

- A negative sequent of length c is a set of $c - 1$ final-$\parr$-free formulas and a distinguished formula A_1, notation $A_1^\perp \vdash A_2, \ldots A_c$. The distinguished formula A_1 may contain final-$\parr$'s.

Rules of MLL$_{\text{nf}}$

$$\frac{}{\vdash \alpha_i, \alpha_i^{\perp}}\text{Ax} \qquad \frac{\vdash \Gamma, A_1, \ldots, A_q}{(\wp(A_1, \ldots, A_q))^{\perp} \vdash \Gamma}\wp; q \geq 1 \qquad \frac{A^{\perp} \vdash \Gamma \quad B^{\perp} \vdash \Delta}{\vdash \Gamma, A \otimes B, \Delta}\otimes$$

The $\wp$-rules stands for $q!C_{q-1}$ different rules, depending on the order of the active formulas in the main formula $\wp(A_1, \ldots, A_q)$ and its actual $\wp$-tree.

A positive (resp. negative) proof with c conclusions is by definition a proof concluding in a positive (resp. negative) sequent of length c.

Equivalence with MLL

Compared to the rules of MLL_{sc}, in MLL_{nf} we are forced to decompose the final-$\parr$'s of the active formulas of a $\otimes$-rule immediately above it, such that we get a final-$\parr$-free sequent again.

This fragment of MLL derives exactly all positive and negative sequents which are MLL-derivable. Indeed, given an MLL proof of a positive or negative sequent, we can successively replace any subderivation

$$
\frac{\vdash \Gamma, A \qquad \frac{\begin{array}{cc} \dots & \dfrac{\vdash \Sigma, C, D}{\vdash \Sigma, C \parr D}\parr \\ \vdots & \vdots \end{array}}{B, \Delta, C \parr D}}{\vdash \Gamma, A \otimes B, \Delta, C \parr D}\otimes
\qquad \text{by} \qquad
\frac{\dfrac{\vdash \Gamma, A \qquad \dfrac{\begin{array}{cc} \dots & \vdash \Sigma, C, D \\ \vdots & \vdots \end{array}}{B, \Delta, C, D}}{\vdash \Gamma, A \otimes B, \Delta, C, D}\otimes}{\vdash \Gamma, A \otimes B, \Delta, C \parr D}\parr
$$

to obtain an MLL_{nf}-proof with the same conclusions (or see [Andreoli 92]).

Splitting tensors

Given a derivable positive sequent $\vdash \Gamma$, we call $A \otimes B \in \Gamma$ a splitting tensor if there is a derivation ending in the introduction of this $\otimes$. Every non-trivial derivable positive sequent has at least one splitting tensor, since it is necessarily introduced by a $\otimes$-rule.

Given a derivable negative sequent $C^{\perp} \vdash \Gamma$, we call $A \otimes B \in \Gamma$ a splitting tensor if there is an $\mathsf{MLL}_{\mathsf{sc}}$-derivation of $\vdash C, \Gamma$ ending in the introduction of this $\otimes$. The distinguished formula C will never be counted as splitting tensor, although there might be an $\mathsf{MLL}_{\mathsf{sc}}$-derivation ending with a $\otimes$-rule with main formula C.

A sequent may have more than one derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{\vdash \alpha_1, \alpha_1^\perp}}{\alpha_1 \vdash \alpha_1}\,\parr
      \qquad
      \dfrac{\dfrac{}{\vdash \alpha_2, \alpha_2^\perp}}{\alpha_2^\perp \vdash \alpha_2^\perp}\,\parr
    }{\vdash \alpha_1, \alpha_1^\perp \otimes \alpha_2, \alpha_2^\perp}\,\otimes
  }{\alpha_2 \vdash \alpha_1, \alpha_1^\perp \otimes \alpha_2}\,\parr
  \qquad
  \dfrac{\dfrac{}{\vdash \alpha_3, \alpha_3^\perp}}{\alpha_3^\perp \vdash \alpha_3^\perp}\,\parr
}{\vdash \alpha_1, \alpha_1^\perp \otimes \alpha_2, \alpha_2^\perp \otimes \alpha_3, \alpha_3^\perp}\,\otimes
$$

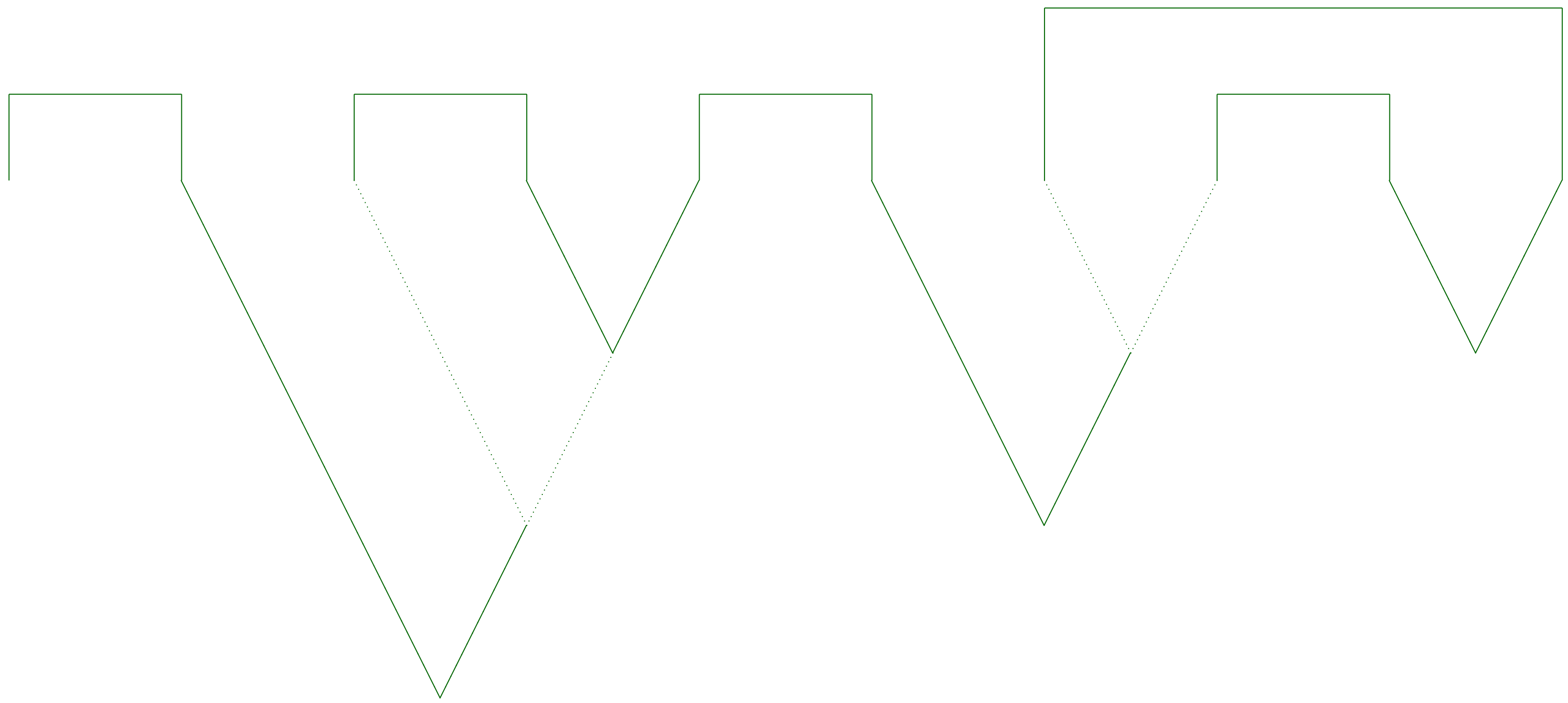
$$
\dfrac{
  \dfrac{\dfrac{}{\vdash \alpha_1, \alpha_1^\perp}}{\alpha_1 \vdash \alpha_1}\,\parr
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{\vdash \alpha_2, \alpha_2^\perp}}{\alpha_2 \vdash \alpha_2}\,\parr
      \qquad
      \dfrac{\dfrac{}{\vdash \alpha_3, \alpha_3^\perp}}{\alpha_3^\perp \vdash \alpha_3^\perp}\,\parr
    }{\vdash \alpha_2, \alpha_2^\perp \otimes \alpha_3, \alpha_3^\perp}\,\otimes
  }{\alpha_2^\perp \vdash \alpha_2^\perp \otimes \alpha_3, \alpha_3^\perp}\,\parr
}{\vdash \alpha_1, \alpha_1^\perp \otimes \alpha_2, \alpha_2^\perp \otimes \alpha_3, \alpha_3^\perp}\,\otimes
$$

A tree-algebra to compute the number of proof nets

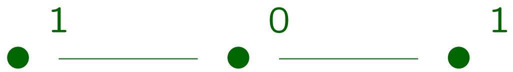
To compute the number of derivable sequents we have to code the geometry of the sequent, by which we mean the geometry of the corresponding proof net. We code the splitting tensor information of a proof net by a labeled tree, called the Splitting Tensor tree (ST-tree).

A labeled tree τ is a tree with edges E and vertices V where to each vertex v is associated a natural number $c_v \geq 0$. It represents a component of a proof net containing c_v conclusions none of which are splitting, while every edge represents a splitting tensor. We denote the set of conclusions by Γ_τ, the cardinality of which equals $|\Gamma_\tau| := |E| + \sum_{v \in V} c_v$. We write $|\tau|$ for the number of edges $|E|$. Specifying some of the components (indicated by an open bullet) we get a so-called specified labeled tree.

Example



has ST-tree



Vector spaces representing proof nets...

Let $\mathbb{P}^{+}$ be the vector space with basic elements labeled trees $[\tau]^{+}$, and $\mathbb{P}^{-}$ be the vector space with basic elements specified labeled trees $[\tau]^{-}$. The element $[\tau]^{+}$ stands for a general proof net with ST-tree τ, while a $[\tau]^{-}$ in which only one component v is specified stands for a general proof net with ST-tree τ, having one conclusion in component v which may contain final-$\parr$'s and is to play an active role in the next $\otimes$-rule.

...turned into algebras

We turn $\mathbb{P}^-$ into an algebra by defining an (associative, commutative) multiplication (denoted by juxtaposition) $\mathbb{P}^- \times \mathbb{P}^- \to \mathbb{P}^-$ by

$$\left[\sigma \ \circ^{\,a} \ \right]^- \left[\ \circ^{\,b} \ \tau\right]^- = \frac{1}{|\sigma| + 1 + |\tau|} \left[\sigma \ \circ^{\,a-1} \!\!\longrightarrow\!\! \circ^{\,b-1} \ \tau\right]^-$$

(intuitively corresponding to applying a $\otimes$-rule to the specified conclusions and remembering the main formula; but taking into account we will encounter this proof net $|\sigma| + 1 + |\tau|$ times);

(two other maps)

Moreover we define a map $\pi : \mathbb{P}^- \longrightarrow \mathbb{P}^+$ by $[\tau]^- \mapsto [\overline{\tau}]^+$ (where $\overline{\tau}$ is obtained by forgetting the specified components of τ, whence corresponding to forgetting the main formula of the $\otimes$-rule) and $\nu : \mathbb{P}^+ \longrightarrow \mathbb{P}^-$ by

$$[\tau]^+ \mapsto \sum_{q=1}^{|\Gamma_\tau|} \left(q! C_{q-1} \sum_{\substack{S \subseteq \Gamma_\tau \\ |S|=q}} [\tau_S]^- \right),$$

corresponding to the $⅋$-rule, i.e. all the possible $⅋$-trees we can attach to a proof net with ST-tree τ. Attaching a q-ary $⅋$-tree to a specific subset S of the conclusions results in a contraction of the subtree spanned by the involved vertices and edges, yielding the contracted specified labeled tree τ_S in which the specified component is the contracted point serving as active component in the $\otimes$-rule.

Example

$$[\bullet \overset{1}{-} \circ \overset{6}{-} \bullet^{1}]^{-}[\circ \overset{4}{-} \bullet^{1}]^{-} = \frac{1}{4}\left[\begin{array}{l} \bullet^{1} \\ \quad \diagdown \\ \qquad \circ \overset{5}{-} \circ \overset{3}{-} \bullet^{1} \\ \quad \diagup \\ \bullet_{1} \end{array}\right]^{-}$$

$$\overset{\pi}{\longmapsto} \quad \frac{1}{4}\left[\begin{array}{l} \bullet^{1} \\ \quad \diagdown \\ \qquad \bullet \overset{5}{-} \bullet \overset{3}{-} \bullet^{1} \\ \quad \diagup \\ \bullet_{1} \end{array}\right]^{+}$$

$$\nu([\bullet \overset{1}{-} \bullet \overset{0}{-} \bullet^{1}]^{+}) = \Big(2[\circ \overset{2}{-} \bullet^{1}]^{-} + 2[\circ \overset{1}{-} \bullet \overset{0}{-} \bullet^{1}]^{-}\Big) +$$

$$+ \Big(4[\circ \overset{1}{-} \bullet^{1}]^{-} + 8[\circ^{3}]^{-}\Big) + 48[\circ^{2}]^{-} + 120[\circ^{1}]^{-}$$

Theorem: number of proof nets

Let us call $S(n, \tau)$ (resp. $T(n, \tau)$) the number of positive (resp. negative) proof nets with n different given axioms and ST-tree τ, i.e. the number of proof nets of type $[\tau]^+$ (resp. $[\tau]^-$).

Theorem 2. *Let us define* $\mathbf{S}(n) := \sum S(n,\tau)[\tau]^+$ *and* $\mathbf{T}(n) := \sum T(n,\tau)[\tau]^-$. *Then the functions* $\mathbf{S} : \mathbb{N} \to \mathbb{P}^+$ *and* $\mathbf{T} : \mathbb{N} \to \mathbb{P}^-$ *satisfy the following recursive definition:*

$$\mathbf{S}(1) = [\ \bullet^{\,2}\ ]^+$$

$$\mathbf{S}(n) = \sum_{n'=1}^{n-1} \binom{n}{n'} \pi(\mathbf{T}(n')\mathbf{T}(n-n')) \qquad (n > 1)$$

$$\mathbf{T}(n) = \nu(\mathbf{S}(n)) \qquad (n \geq 1)$$

The number of derivations; an upperbound

In order to compute an upperbound, we may consider a modified version of the product of two base elements in the ST-algebra, in which we don't keep track of the number of times we encounter the corresponding proof net.

If we don't keep track of the number of times we encounter a proof net we arrive at functions counting the number of positive (resp. negative) derivations, thus providing an upperbound for the number of derivable suquents.

So let us modify the multiplication of $\mathbb{P}^{-}$ into

$$[\sigma \circ^{a}\ ]^{-}[\ \circ^{b}\ \tau]^{-} = [\sigma \circ^{a-1} \rule[0.5ex]{3em}{0.4pt} \circ^{b-1}\ \tau]^{-}$$

(intuitively corresponding to applying a $\otimes$-rule to the specified conclusions and remembering the main formula).

Then the previous theorem holds again for the number of derivations.

Only the number of conclusions matters

In the previous computation the geometry of the sequent does not play a role anymore, whence may be forgotten. Replacing each ST-tree by the number of its conclusions, we obtain the following description of the algebra and generating functions.

Let $\mathbb{P}^+$ and $\mathbb{P}^-$ be two isomorphic copies of the countably-dimensional vector space $\mathbb{R}^\omega := \left\{\sum_{i=1}^n c_i \mathbf{e}_i \;\middle|\; c_i \in \mathbb{R}, n \in \mathbb{N}\right\}$. The basic elements of $\mathbb{P}^+$ will be written $[i]^+$ (standing for a general positive proof with i conclusions), and those of $\mathbb{P}^-$ will be written $[i]^-$ (standing for a general negative proof with i conclusions). Let us define $\mathbf{P}(n) := \sum_{c\geq 1} P(n,c)[c]^+$ and $\mathbf{Q}(n) := \sum_{c\geq 1} Q(n,c)[c]^-$.

(multiplication)

We turn $\mathbb{P}^-$ into an algebra by defining an (associative, commutative) multiplication (denoted by juxtaposition) $\mathbb{P}^- \times \mathbb{P}^- \longrightarrow \mathbb{P}^-$ by

$$[i]^-[j]^- = [i + j - 1]^-$$

intuitively corresponding to applying a $\otimes$-rule and remembering the main formula:

$$\frac{A^\perp \vdash \Gamma \quad B^\perp \vdash \Delta}{(A \otimes B)^\perp \vdash \Gamma, \Delta}$$

(two other maps)

Moreover we define a linear map $\pi : \mathbb{P}^- \longrightarrow \mathbb{P}^+$ by $[i]^- \mapsto [i]^+$, forgetting the distinguished formula

$$\frac{A^\perp \vdash \Gamma}{\vdash A, \Gamma}$$

And finally we define $\nu : \mathbb{P}^+ \longrightarrow \mathbb{P}^-$ by

$$[i]^+ \mapsto \sum_{q=1}^{i} \binom{i}{q} q! C_{q-1} [i - q + 1]^-$$

corresponding to the $⅋$-rule, i.e. all the possible $⅋$-trees we can attach to a positive sequent of length i.

Theorem: number of derivations

Theorem 3. *The functions* $\mathbf{P} : \mathbb{N} \to \mathbb{P}^{+}$ *and* $\mathbf{Q} : \mathbb{N} \to \mathbb{P}^{-}$ *satisfy the following recursive definition:*

$$\mathbf{P}(1) = [2]^{+}$$

$$\mathbf{P}(n) = \sum_{n'=1}^{n-1} \binom{n}{n'} \pi(\mathbf{Q}(n')\mathbf{Q}(n-n')) \qquad (n > 1)$$

$$\mathbf{Q}(n) = \nu(\mathbf{P}(n)) \qquad (n \geq 1)$$

Explicit formula for the coefficients

Theorem 4.

$$P(1,1) = 0$$

$$P(1,2) = 1$$

$$P(n,c) = \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{\substack{c'=0 \\ n'-((n+1)-c)\leq c'\leq n'}}^{c-1} Q(n', 1+c')Q(n-n', 1+((c-1)-c'))$$

$$(n > 1; 1 \leq c \leq n+1)$$

$$Q(n,c) = \sum_{q=1}^{1+((n+1)-c)} \binom{q+(c-1)}{q} q! C_{q-1} P(n, q+(c-1))$$

$$(n \geq 1; 1 \leq c \leq n+1)$$

Results

n	number of provable formulas $T(n,1)/(2^n n!)$	number of $\mathsf{MLL}_{\mathsf{nf}}$-derivations $Q(n,1)/(2^n n!)$	number of balanced formulas $\binom{2n-1}{n} C_{2n-1} (2n)!/(2^n n!)$
1	1	1	1
2	17	17	45
3	882	1 174	6 300
4	82 725	174 213	1 576 575
5	11 556 590	43 508 186	578 918 340
6	2,173 613 962	16,093 558 826	282,319 177 140
7	517,553 880 484	8 162,702 679 852	172 272,314 214 000
8	149 714,681 114 349	5 394 878,462 002 605	126 458 645,652 714 375
9	51 094 054,734 001 494	4,482 152 731,426 496 050	108,604 558 347,968 182 500
10	20,126 763 226,141 651 806	4 558,136 970 068,451 778 302	106 888,606 326 070,285 216 500

A heuristic lowerbound based on the number of dervs

Suppose we know the function $T(n,c,s)$ of $\mathsf{MLL}_{\mathsf{nf}}$-derivable negative sequents of length c with n axioms and with $s \geq 0$ splitting tensors among the non-distinguished $c-1$ final-⅋-free conclusions. Then the number $S(n,c)$ of derivable positive sequents of length c on n axioms would be

$$S(1,1) = 0$$

$$S(1,2) = 1$$

$$S(n,c) = \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} \sum_{s_1 \geq 0} \sum_{s_2 \geq 0} \frac{T(n', 1+c', s_1)T(n-n', c-c', s_2)}{s_1 + 1 + s_2}$$

$$(n > 1; 1 \leq c \leq n+1)$$

which is similar to the expression for $P(n,c)$ except for the fact that for each pair (s_1, s_2) we divide the summand by $s = s_1 + 1 + s_2$ in order to correct for sequents counted several times.

Expectation

Let D be a non-empty finite set (called domain) and X a real-valued function on D. The expectation of X is given by

$$\mathbb{E}X := \frac{1}{|D|}\sum_{d\in D} X(d) = \frac{1}{|D|}\sum_{x} n_x x$$

where n_x is the frequency of the value x:

$$n_x := |\{d \in D \mid X(d) = x\}| = |X^{-1}(\{x\})|$$

The operation $\mathbb{E}$ is linear ($\mathbb{E}(aX + bY) = a\mathbb{E}X + b\mathbb{E}Y$) but $\mathbb{E}(XY) = (\mathbb{E}X)(\mathbb{E}Y)$ in general does not hold; the left-hand-side may be either less or greater than the right-hand-side.

E.g. if $D = \{a, b\}$; $X(a) = 0$; $X(b) = 1$; $Y = 1 - X$ then $\mathbb{E}(XY) = 0$ while $\mathbb{E}X\mathbb{E}Y = (\frac{1}{2})^2$. Also, $\mathbb{E}(XX) = \frac{1}{2}$ while $\mathbb{E}X\mathbb{E}X = (\frac{1}{2})^2$.

Lemma

Lemma 5. *Suppose X is positive. Then*

$$\mathbb{E}\left(\frac{1}{X}\right) \geq \frac{1}{\mathbb{E}X}.$$

We will use

Let $z > 0$, then we can take its root. As for every square, $(\sqrt{z} - \frac{1}{\sqrt{z}})^2 \geq 0$, i.e. $z - 2\sqrt{z}\frac{1}{\sqrt{z}} + \frac{1}{z} \geq 0$, and we conclude that $z + \frac{1}{z} \geq 2$.

Proof

$$\mathbb{E}\left(\frac{1}{X}\right)\mathbb{E}X = \left(\frac{1}{|D|}\sum_x n_x \frac{1}{x}\right)\left(\frac{1}{|D|}\sum_y n_y y\right) = \frac{1}{|D|^2}\sum_{x,y} n_x n_y \frac{y}{x} =$$

$$= \frac{1}{|D|^2}\left(\sum_{x<y} n_x n_y \left(\frac{y}{x}+\frac{x}{y}\right) + \sum_{x=y} n_x n_y \frac{y}{x}\right) \geq$$

$$\geq \frac{1}{|D|^2}\left(\sum_{x<y} 2 n_x n_y + \sum_{x=y} n_x^2\right) =$$

$$= \frac{1}{|D|^2}\left(\sum_x n_x\right)\left(\sum_y n_y\right) = 1$$

where we used the fact that for positive y and x it holds that $\frac{y}{x}+\frac{x}{y}\geq 2$.

The expected number of splitting tensors

However, for each splitting tensor we could count the number of cases in which this specific tensor remains splitting after applying a $\wp$-rule (the attaching of a $\wp$-tree); an idea which we will explore now.

We can rewrite the expression for $S(n,c)$ (the number of derivable sequents of length c on n axioms) as follows:

$$S(1,1) = 0$$

$$S(1,2) = 1$$

$$S(n,c) = \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} \sum_{s_1 \geq 0} \sum_{s_2 \geq 0} \frac{T(n',1+c',s_1)T(n-n',c-c',s_2)}{s_1+1+s_2} =$$

$$= \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} T(n',1+c')T(n-n',c-c')\mathbb{E}\left(\frac{1}{X(n',1+c')+1+X(n-n',c-c')}\right)$$

$$(n > 1; 1 \leq c \leq n+1)$$

(definition $X(n,c)$)

where the integer-valued function $X(n,c) : D(n,c) \to \mathbb{N}$ is the number of splitting tensors; a function on the set $D(n,c)$ of derivable sequents of length c with n axioms and one distinguished formula (to play an active role in $\otimes$ and allowed to have final-$\parr$'s) that counts the number of splitting tensors among the remaining $c-1$ final-$\parr$-free conclusions ($0 \leq X(n,c) \leq c-1$). Observe that $T(n,c) := |D(n,c)| = \sum_s T(n,c,s)$ satisfies

$$T(n,c) = \sum_{q \geq 1} \binom{q+(c-1)}{q} q! C_{q-1} S(n, q+(c-1))$$

(cf. the formula for $Q(n,c)$).

Probability that tensor remains splitting

Let us consider one summand

$$\binom{c}{q} q! C_{q-1} S(n, c)$$

of the expression for $T(n, \tilde{c}) = \sum_{q \geq 1} \binom{q+(\tilde{c}-1)}{q} q! C_{q-1} S(n, q + (\tilde{c} - 1))$ (so $c = q + (\tilde{c} - 1)$).

$$\dfrac{\dfrac{\dfrac{\vdash \Delta, \Sigma}{(\parr(\Sigma))^{\perp} \vdash \Delta}\parr \quad \dfrac{\vdash \Sigma', \Delta'}{(\parr(\Sigma'))^{\perp} \vdash \Delta'}\parr}{\vdash \Delta, \parr(\Sigma) \otimes \parr(\Sigma'), \Delta'}\otimes}{\dfrac{\vdash \Gamma, \Pi}{(\parr(\Pi))^{\perp} \vdash \Gamma}\parr}\ \text{Id}$$

Attaching a $⅋$-tree

Given a non-trivial proof π with $|\Gamma, \Pi| = c$ final-$⅋$-free conclusions, the final $\otimes$-rule defines a partition of the context conclusions in $|\Delta| = c'$ and $|\Delta'| = (c-1) - c'$ conclusions. Attaching a $⅋$-tree with q leaves ($⅋(\Pi)$) results in a sequent with $\tilde{c} := c - q + 1$ conclusions (so $c = q + (\tilde{c} - 1)$) where this $\otimes$ is still splitting iff it is completely applied within Δ or Δ', i.e. in $\left(\binom{c'}{q} + \binom{(c-1)-c'}{q}\right) q! C_{q-1}$ of the $\binom{c}{q} q! C_{q-1}$ cases.

Expected number of splitting tensors given q

As said, in the formula for $P(n,c)$ we consider every sequent s times, viz. once for every splitting tensor. Hence the expected number of splitting tensors $E(n,c,q)$ among the $\binom{c}{q} q! C_{q-1} S(n,c)$ derivable sequents on n axioms and with $\tilde{c}$ conclusions, one of which being a q-ary $\parr$-tree which is to play an active role in the next $\otimes$-rule, is given by

$$E(n,c,q) := \frac{1}{S(n,c)} \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} T(n', 1+c') T(n-n', c-c') \frac{\binom{c'}{q} + \binom{(c-1)-c'}{q}}{\binom{c}{q}}$$

in which, for a given q, every summand is multiplied by $\frac{\binom{c'}{q}+\binom{(c-1)-c'}{q}}{\binom{c}{q}}$.

Expected number of splitting tensors

This in turn can be applied to the above formula for $T(n,c)$ to get the expected number of splitting tensors $\mathbb{E}X(n,c)$ in each of the branches of the $\otimes'$-rule.

$$\mathbb{E}X(n,c) = \frac{1}{T(n,c)} \sum_{q\geq 1} \binom{q+(c-1)}{q} q! C_{q-1} S(n, q+(c-1)) E(n, q+(c-1), q) =$$

$$= \frac{1}{T(n,c)} \sum_{q\geq 1} q! C_{q-1} \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c'\geq 0} T(n', 1+c') T(n-n', q+(c-1)-c') \left(\binom{c'}{q} + \binom{(q+(c-1)-1)-c'}{q} \right)$$

The final step

We found a perfect expression for $\mathbb{E}X(n,c)$, but the inductive formula for $S(n,c)$ depends on $\mathbb{E}\left(\frac{1}{X(n',1+c')+1+X(n-n',c-c')}\right)$. In fact, by $\mathbb{E}\left(\frac{1}{X}\right) \geq \frac{1}{\mathbb{E}X}$ we know that for $n > 1, 1 \leq c \leq n+1$:

$$S(n,c) = \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} T(n',1+c')T(n-n',c-c')\mathbb{E}\left(\frac{1}{X(n',1+c')+1+X(n-n',c-c')}\right) \geq$$

$$\geq \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} T(n',1+c')T(n-n',c-c')\frac{1}{\mathbb{E}\left(X(n',1+c')+1+X(n-n',c-c')\right)} =$$

$$= \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} \frac{T(n',1+c')T(n-n',c-c')}{\mathbb{E}X(n',1+c')+1+\mathbb{E}X(n-n',c-c')}$$

Heuristic approximation

This leads to the following heuristic approximations S', F' and T' for S, $\mathbb{E}X$ respectively T, defined by an easy recursion:

$$S'(1,1) = 0$$

$$S'(1,2) = 1$$

$$S'(n,c) = \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} \frac{T'(n', 1+c')T'(n-n', c-c')}{F'(n', 1+c') + 1 + F'(n-n', c-c')}$$

$$(n > 1; 1 \leq c \leq n+1)$$

$$T'(n,c) = \sum_{q \geq 1} \binom{q + (c-1)}{q} q! C_{q-1} S'(n, q + (c-1))$$

$$(n \geq 1; 1 \leq c \leq n+1)$$

$$F'(n,c) = \frac{1}{T'(n,c)} \sum_{q \geq 1} q! C_{q-1} \sum_{n'=1}^{n-1} \binom{n}{n'} \sum_{c' \geq 0} T'(n', 1+c')T'(n-n', q+(c-1)-c') \left(\binom{c'}{q} + \binom{(q+(c-1)-1)-c'}{q} \right)$$

$$(n \geq 1; 1 \leq c \leq n+1)$$

Results

n	number of provable formulas $T(n,1)/(2^n n!)$	heuristic approximation $T'(n,1)/(2^n n!)$	number of $\mathsf{MLL}_{\mathsf{nf}}$-derivations $Q(n,1)/(2^n n!)$
1	1	1.0	1
2	17	17.0	17
3	882	810.5	1,174
4	82,725	67,180.8	174,213
5	11,556,590	8,097,633.2	43,508,186
6	2;173,613,962	1;292,177,393.4	16;093,558,826
7	517;553,880,484	257;683,716,149.8	8,162;702,679,852
8	149,714;681,114,349	61,774;586,215,171.7	5,394,878;462,002,605
9	51,094,054;734,001,494	17,316,387;694,269,184.1	4;482,152,731;426,496,050
10	20;126,763,226;141,651,806	5;559,590,039;485,795,1xx.x	4,558;136,970,068;451,778,302

Conclusion and remarks

- The heuristic lowerbound yields an efficient and accurate approximation to the exact number of provable formulas, which is unfeasible for Marco's laptop when $n > 10$;

- The expectation formula provides the technical point used to overcome the difficult question on how splitting tensors influence the possible structures

- It seems mandatory to use the geometric structure, in order to compute the number of proof nets, since we succeded into giving up the information coming with the ST-algebra, only in the case of derivations;

- In all the different calculi we presented here, weakening can be compatibly included with our method; in particular, when considering the completely focalized calculus we get close to ludics (counting can be yet another, indirect, motivation for focalized calculi);

- The fragment we didn't considered at all, is the calculus with additive connectives, analog results for MALL seem to be very hard to be obtained;

- A primary goal for our investigation is to compare the combinatorial properties of syntax with respect to combinarial properties of semantics: a future direction of investigation will be to consider similar questions in denotational semantics (coherence spaces);

- Probabilistic correctness criterion...