Paths: from Lévy's labels to the geometry of interaction
A retrospective and a few remarks*

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Optimality (Lévy)   Asperti-Laneve (AL)

Gonthier-Abadi-Lévy

GOI (Girard)   Danos-Regnier (DR)

The equivalences of paths (historically)

$$\text{virtual} \overset{\text{AL}}{\Leftrightarrow} \text{legal} \overset{\text{ADLR}}{\Leftrightarrow} \text{regular} \overset{\text{DR}}{\Leftrightarrow} \text{persistent}$$

$$\text{regular} \Updownarrow_{\text{AB}^*} \text{consistent}$$

Remark: persistent and virtual are "morally" the same

A path through path equivalences

		persistent		
		$\Updownarrow$		
	$\Leftarrow$	s-persistent	$\Leftrightarrow$	regular
virtual		$\Uparrow$		$\Updownarrow$
	$\Rightarrow$	legal		consistent

A bijective correspondence*

virtual $\leftrightarrow$ extraction normal form

$$\phi \mapsto \tilde{\phi}$$

where **extraction** (Lévy) progressively tames redexes that do not contribute to the creation of the last redex of the derivation.

Extended residual theory

Residuals of

paths	Regnier
extraction normal forms	Melliès

Claim*: the correspondence $\phi \mapsto \tilde{\phi}$ commutes with residuals:

$\phi /_u \psi$ iff $\tilde{\phi} =$ normal form of $u; \tilde{\psi}$.

Residuals of paths

Just as redexes, paths can be duplicated or erased through reduction, but in addition they can also be **broken**.

Persistent

persistent: a path ϕ that no reduction can break (there does not exist $(D;u)$, ψ s.t. $\phi/_D\,\psi$ and u breaks ψ).

s-persistent *: a path that traverses no redex, or a path ϕ s.t. $\boldsymbol{\phi}/\boldsymbol{u}$ is s-persistent, where $\boldsymbol{u}$ is the leftmost-outermost redex traversed by $\boldsymbol{\phi}$.

s-persistent $\overset{\text{known to Regnier}}{\Rightarrow}$ persistent

Lemma*: if ϕ is broken by D_1 and if $D_1 \equiv D_2$, then also D_2 breaks ϕ.

Let ϕ be s-persistent, and suppose D breaks ϕ. Let D' be the derivation that reduces repetitively the leftmost-outermost redex of (the residual of) ϕ. Complete into $D; D'' \equiv D'; D'''$. Then $D; D''$ breaks ϕ while $D'; D'''$ erases it, contradicting the lemma.

Virtual

[VIRTUAL-I] A @ $-\,\lambda$-path ϕ s.t. $\exists\, N \;\; M \rightarrow^{*} N$ and $N = C[(\lambda x.P)^{l} Q]$ with $\phi = path(l)$.

[VIRTUAL-II]* A redex $\boldsymbol{u}$, or a @$\boldsymbol{\lambda}$-path $\boldsymbol{\phi}$ s.t. $\boldsymbol{\phi}/\boldsymbol{u}$ is virtual, for some / all redex $\boldsymbol{u}$ traversed by $\boldsymbol{\phi}$.

$$\begin{array}{ccc} \text{virtual-II} & \Leftarrow & \text{s-persistent} \\ \Downarrow & & \Uparrow \\ \text{virtual-I} & \Rightarrow & \text{legal} \end{array}$$

The implication virtual-I $\Rightarrow$ legal is shown by AL, who also implicitly follow the above factorization of the reverse implication.

Lévy's labels

Labels: $\alpha ::= a | \alpha_1 \alpha_2 | \overline{\alpha} | \underline{\alpha}$

Terms: $\begin{cases} M ::= T^{\alpha} \\ T ::= x | M_1 M_2 | \lambda x.M \end{cases}$

$$((\lambda x.M)^{\alpha} N)^{\beta} \to (\beta \overline{\alpha}) \cdot (M[x \leftarrow \underline{\alpha} \cdot N])$$

with $\beta \cdot T^{\alpha} = T^{\beta\alpha}$, $x^{\alpha}[x \leftarrow M] = \alpha \cdot M$

(and $T^{\alpha}[x \leftarrow M] = (T^{\alpha}[x \leftarrow M])^{\alpha}$ otherwise)

Labels induce paths

(Asperti-Laneve)

$$path(a) = a \qquad path(\alpha\beta) = path(\alpha)path(\beta)$$

$$path(\overline{\alpha}) = path(\alpha) \quad path(\underline{\alpha}) = path(\alpha)^*$$

where ϕ^* is the reverse of ϕ.

Legal (AL)

A **well-balanced path** (wbp) is either a redex, or a @λ-wbp (or reverse @λ-wbp) suitably preceded by a wbp and followed by a suitable edge.

A wbp is **legal** if all suitable cycles are accessed and left by suitable paths which are reverse of each other.

Geometry of interaction

Associate with each edge a (partial, invertible) operator. Then to each path associate the corresponding compound operator. If the operator is non-zero (i.e., its domain of definition is non-empty), then the path is accepted by the "machine".

Free GOI

$$\mathsf{u} ::= \mathsf{k}|0|1|\mathsf{u}_1\mathsf{u}_2|\mathsf{u}^*|!(\mathsf{u}) \qquad \mathsf{u}0 \longrightarrow 0$$
$$\mathsf{k} ::= \mathsf{m}|\mathsf{e} \qquad 0\mathsf{u} \longrightarrow 0$$
$$\mathsf{m} ::= p|q \qquad \mathsf{u}1 \longrightarrow \mathsf{u}$$
$$\mathsf{e} ::= r|s|t|d \qquad 1\mathsf{u} \longrightarrow 1$$

$$(\mathsf{u}_1\mathsf{u}_2)^* \longrightarrow \mathsf{u}_2^*\mathsf{u}_2^*$$
$$!(\mathsf{u}_1\mathsf{u}_2) \longrightarrow (!(\mathsf{u}_1))(!(\mathsf{u}_2))$$
$$(!(\mathsf{u}))^* \longrightarrow\ !(\mathsf{u}^*)$$

The interesting equations

$$\mathsf{k}^*\mathsf{k} \longrightarrow 1$$

$$\mathsf{m}_1^*\mathsf{m}_2 \longrightarrow 0 \quad (\mathsf{m}_1 \neq \mathsf{m}_2)$$

$$\mathsf{e}_1^*\mathsf{e}_2 \longrightarrow 0 \quad (\mathsf{e}_1 \neq \mathsf{e}_2)$$

$$(!(\mathsf{u}))\mathsf{e} \longrightarrow \mathsf{e}(!^{c(\mathsf{e})}(\mathsf{u}))$$

$$(\mathsf{e}^*)(!(\mathsf{u})) \longrightarrow (!^{c(\mathsf{e})}(\mathsf{u}))(\mathsf{e}^*)$$

$$(c(\mathsf{r}) = c(\mathsf{s}) = 1, c(\mathsf{t}) = 2, c(\mathsf{d}) = 0)$$

Weights

$$[\![x]\!] = x^{\mathsf{d}}$$
$$[\![\lambda x.M]\!] = (\lambda x.\mathsf{q} \cdot [\![M]\!][x \leftarrow x \cdot \mathsf{p}])^{1}$$
$$[\![MN]\!] = (([\![M]\!] \cdot \mathsf{r})(\mathsf{p} \cdot !([\![N]\!]) \cdot \mathsf{t} \cdot \mathsf{s})))^{\mathsf{q}}$$

with $\begin{cases} k \cdot T^{\mathsf{u}} = T^{\mathsf{uk}} \quad !(T^{\mathsf{u}}) = !(T)^{!(\mathsf{u})} \\ P \cdot \mathsf{k} = P[\vec{x} \leftarrow x \cdot \vec{\mathsf{k}}] \\ x^{\mathsf{u}}[x \leftarrow x \cdot \mathsf{k}] = x^{\mathsf{ku}} \end{cases}$

(the translation also embodies the underlying translation into proof-nets)

Regular

A regular path is a path of non provably null weight. This property is decidable, thanks to Regner's AB* theorem, of which we present a "**sorted**" / "locative" version*.

AB^* **theorem**

l The above rewriting system brings any

$$\left.\begin{array}{r}\text{non provably null}\\ \text{provably null}\end{array}\right\}\textit{weight of a path}$$

$$\text{to}\begin{cases}0\\ \text{the form } A(B^*)\end{cases}$$

where A and B are ***-free**.

Sorts

$T ::= m|m(T,T)$ (m natural number)

where m records the box depth and branching records the locative structure of multiplicatives. For example, $1(2,0)$ " comes", say, from $?(!!A \otimes B)$.

Lifting operation: $\begin{cases} L(m) = m+1 \\ L(m(S,T)) = (m+1)(S,T) \end{cases}$

Sorting rules

$$\mathsf{p} : S \to \mathsf{0}(S, T)$$
$$\mathsf{q} : T \to \mathsf{0}(S, T)$$
$$\mathsf{r} : L(S) \to L(S)$$
$$\mathsf{s} : L(S) \to L(S)$$
$$\mathsf{d} : S \to L(S)$$
$$\mathsf{t} : L(L(S)) \to L(S)$$

$$\frac{\mathsf{v} : S \to T \quad \mathsf{w} : T \to U}{\mathsf{wv} : S \to U}$$

$$\frac{\mathsf{v} : S \to T}{\mathsf{v}^* : T \to U}$$

$$\frac{\mathsf{v} : S \to T}{!(\mathsf{v}) : L(S) \to L(T)}$$

Sorted AB* theorem*

Every well-sorted u rewrites either to 0 or to an AB* form.

Proof: 1) Subject reduction:

u : $S \to T$ and u $\to$ v $\Rightarrow$ v : $S \to T$

2) Every well-sorted normal form is 0 or in AB* form;

Any *weight of a path* is well-sorted, hence original AB* follows.

All non-trivial models of GOI induce the same paths

This is a consequence of AB^*.

Suppose, for u weight of a path, that $\mathcal{M} \models \mathsf{u} = 0$ and that $\nvdash \mathsf{u} = 0$. Then $\vdash \mathsf{u} = \mathsf{AB}^*$ for some *-free A, B, hence $\mathcal{M} \models \mathsf{AB}^* = 0$, which entails

$$1 = \mathsf{A}^*\mathsf{AB}^*\mathsf{B} = \mathsf{A}^*0\mathsf{B} = 0$$

In particular, regular=consistent.

Consistent

The context semantics (or token machine) acts on t-uples of stacks.

p pushes p on the rightmost stack
(idem for q, r, s)
d opens a new stack initialized with □
t contracts the rightmost two stacks s, t as a single stack
!(u) leaves the rightmost stack intact
and lets u act on the remaining ones

The above sorts might be useful here.

Regular $\Leftrightarrow$ s-persistent

This follows from:

1) If a path ϕ traverses at least a redex, and if u is the leftmost-outermost one, then ϕ is regular iff the reduction of u does not break ϕ and ϕ/u is regular (note that ϕ/u is shorter).

2) If a path ϕ traverses no redex, then its weight is of the form AB*, hence it is regular.